
Supplement for Fast Scalable and Accurate Discovery of DAGs Using the Best Order Score Search and Grow-Shrink Trees

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1 Simulations

Linear Gaussian simulations are included to establish the performance of BOSS compared to existing causal discovery algorithms on Erdős-Rényi and scale-free graphs. These results are tabulated in Tables 1, 2, 3, and 4 which compare the following algorithms:

- BOSS
- GRaSP
- fGES
- PC
- DAGMA - converted to CPDAG

LiNGAM was not included in the linear Gaussian comparison because no non-Gaussian signal is available. Linear Non-Gaussian simulations are included to establish the performance of BOSS compared to methods that (can) take advantage of non-Gaussian signal. These results are tabulated in Tables 5, 6, 7, and 8 which compare the following algorithms:

- BOSS
- DAGMA
- LiNGAM

High dimensional simulations are included to establish the scalability of BOSS compared to GRaSP. These results are tabulated in Tables 9 and 10.

1.1 Algorithms

In what follows, we give a complete list of the tuning parameter settings we used for all algorithms:

- BOSS - Best Order Score Search
 - SemBicScore()
 - * setPenaltyDiscount(2/4/8)
 - * setStructurePrior(0)
 - * setRuleType(SemBicScore.RuleType.CHICKERING)
 - setUseBes(False)
 - setNumStarts(1)
 - setNumThreads(1)
 - setDataOrder(false)
- GRaSP - Greedy Relaxations of the Sparsest Permutation [5, 7]
 - SemBicScore()
 - * setPenaltyDiscount(2/4/8)
 - * setStructurePrior(0)
 - * setRuleType(SemBicScore.RuleType.CHICKERING)
 - setDepth(3)
 - setNonSingularDepth(1)
 - setUncoveredDepth(1)
 - setNumStarts(1)
 - setOrdered(false)
 - setDataOrder(false)
- fGES - fast Greedy Equivalent Search [3, 6, 7]
 - SemBicScore()
 - * setPenaltyDiscount(2)
 - * setStructurePrior(0)
 - * setRuleType(SemBicScore.RuleType.CHICKERING)
 - setFaithfulnessAssumed(false)
 - setMaxDegree(-1)
- PC - Peter and Clark [7, 9]
 - ScoreIndTest(SemBicScore())
 - * setPenaltyDiscount(2)
 - * setStructurePrior(0)
 - * setRuleType(SemBicScore.RuleType.CHICKERING)
 - setDepth(1000)
 - setStable(false)
 - setConflictRule(ConflictRule.OVERWRITE_EXISTING)
 - setAggressivelyPreventCycles(false)
- DAGMA - DAGs via M-matrices for Acyclicity [2]
 - loss_type = 'l2'
 - lambda1 = 0.1
 - w_threshold = 0.1
 - T = 4
 - mu_init = 1.0
 - mu_factor = 0.1
 - s = [1.0, .9, .8, .7]
 - warm_iter = 2e4
 - max_iter = 7e4
 - lr = 0.0003

- checkpoint = 1000
- beta_1 = 0.99
- beta_2 = 0.999
- LiNGAM - Linear Non-Gaussian Acyclic Model [4, 8]
 - random_state = None
 - prior_knowledge = None
 - apply_prior_knowledge_softly = False
 - measure = 'pwing'

1.2 Data-generation

Erdős-Rényi and scale-free graphs were generated according to Algorithms 2 and 3, respectively. For scale-free graphs, the parents of an Erdős-Rényi graph are redrawn to produce a graph whose out-degree follows a power-law. The redrawing process follows a modified Barabási-Albert model where the preferential attachment of each potential parent is inflated by one to account for vertices with zero out-degree [1]. Figure 1 depicts the in/out degree distributions for scale-free graphs (100 vertices, average degree 10) where the solid black line gives the median and the grayed region denotes an empirical 95% confidence interval. Notably, the in/out-degree for Erdős-Rényi graphs will both follow the in-degree distribution defined in Figure 1.

Data were generated according to Algorithm 1 where \mathcal{U} denotes a uniform distribution. In our simulations, the error \mathcal{E} is either distributed as Gaussian, Gumbel, or Exponential. In all cases, the columns of the data matrix were shuffled prior to being passed to a search algorithm, so that the variable order in the dataset does not match the order of data generation.

Algorithm 1: $\text{simulate}(\mathcal{G}, \mathcal{E}, n)$

Input: DAG : $\mathcal{G} = (V, E)$ error : \mathcal{E} samples : n

Output: data : \mathbf{X}

```

foreach  $v \in V$  do
   $\sigma \sim \mathcal{U}(1, 2)$ 
  foreach  $w \in \text{pa}_{\mathcal{G}}(v)$  do
     $\beta \sim \mathcal{U}(-1, 1)$ 
    foreach  $i \in [n]$  do
       $\mathbf{X}_{i,v} \sim \mathcal{E}(\sigma)$ 
       $\mathbf{X}_{i,v} \leftarrow \mathbf{X}_{i,v} + \beta \mathbf{X}_{i,w}$ 
   $\mathbf{X}_v \leftarrow \text{standardize}(\mathbf{X}_v)$ 

```

Algorithm 2: $\text{ER-DAG}(V, \pi, \alpha)$

Input: vertices : V perm : π avg deg : α

Output: DAG : $\mathcal{G} = (V, E)$

$E \leftarrow \emptyset$

repeat

```

   $(v, w) \sim f : V \times V \rightarrow \mathbb{R}$  s.t.  $f \propto 1_{\text{pre}_{\pi}(v)}(w)$ 
   $E \leftarrow E \cup \{(v, w)\}$ 

```

until $|E| = \frac{\alpha}{2} |V|$

Algorithm 3: SF-DAG(V, π, α)

Input: vertices : V perm : π avg deg : α **Output:** DAG : $\mathcal{G} = (V, E)$ $\mathcal{H} \leftarrow \text{ER-DAG}(V, \pi, \alpha)$ $E \leftarrow \emptyset$ **for** $v \in \pi$ **do** **repeat** $w \sim f : V \rightarrow \mathbb{R}$ s.t. $f \propto 1_{\text{pre}_\pi(v)}(w) + |\text{ch}_{\mathcal{G}}(w)|$ $E \leftarrow E \cup \{(v, w)\}$ **until** $|\text{pa}_{\mathcal{G}}(v)| = |\text{pa}_{\mathcal{H}}(v)|$

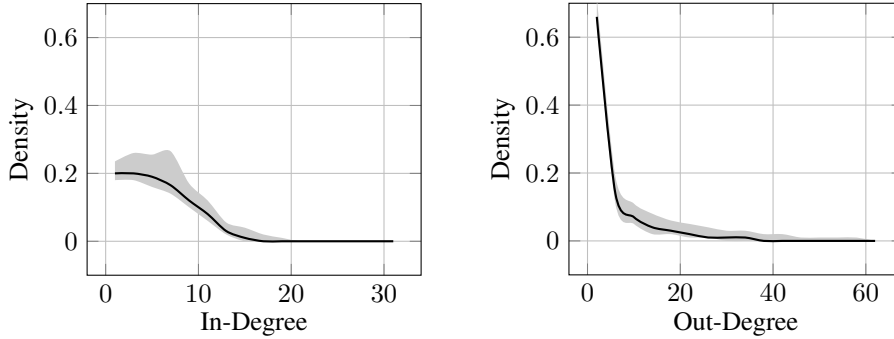


Figure 1: In/out-degree distributions for scale-free simulations with 100 variables and average degree 10 over 100 repetitions.

2 Complete Results

Tables 1 and 2 tabulate results for BOSS, GRaSP, fGES, PC, and DAGMA run on linear Gaussian models with Erdős-Rényi graphs, 100 variables, and sample size 1000, for average degrees 2 through 20. Table 1 shows adjacency precision, adjacency recall, orientation precision, and orientation recall. Table 2 shows the BIC score difference compared to the true graph, number of edges in the estimated graph, and elapsed wall time in seconds. Mean statistics over 20 repetitions are shown with standard deviations in parentheses.

Tables 3 and 4 tabulate results for BOSS, GRaSP, fGES, PC, and DAGMA run on linear Gaussian models with scale-free graphs, 100 variables, and sample size 1000, for average degrees 2 through 20. Table 3 shows adjacency precision, adjacency recall, orientation precision, and orientation recall. Table 4 shows the BIC score difference compared to the true graph, number of edges in the estimated graph, and elapsed wall time in seconds. Mean statistics over 20 repetitions are shown with standard deviations in parentheses.

Tables 5 and 6 tabulate results for BOSS, DAGMA, and LiNGAM run on linear Gumbel models with scale-free graphs, 100 variables, and sample size 1000, for average degrees 2 through 20. Table 5 shows adjacency precision, adjacency recall, orientation precision, and orientation recall. Table 6 shows the BIC score difference compared to the true graph, number of edges in the estimated graph, and elapsed wall time in seconds. Mean statistics over 20 repetitions are shown with standard deviations in parentheses.

Tables 7 and 8 tabulate results for BOSS, DAGMA, and LiNGAM run on linear Exponential models with scale-free graphs, 100 variables, and sample size 1000, for average degrees 2 through 20. Table 7 shows adjacency precision, adjacency recall, orientation precision, and orientation recall. Table 8 shows the BIC score difference compared to the true graph, number of edges in the estimated graph, and elapsed wall time in seconds. Mean statistics over 20 repetitions are shown with standard deviations in parentheses.

Tables 9, 10, 11 and 12 tabulate results for BOSS and GRaSP run on linear Gaussian models with scale-free graphs, average degree 20, and sample size 1000, for numbers of variables 100 through

1000; runs exceeding two hours were cancelled and are not reported. Tables 9 and 10 show adjacency precision, adjacency recall, orientation precision, and orientation recall. Tables 11 and 12 show the BIC score difference compared to the true graph, number of edges in the estimated graph, and elapsed wall time in seconds. Mean statistics over 20 repetitions are shown with standard deviations in parentheses.

Tables 13 and 14 tabulate results for BOSS, DAGMA, fGES, and LiNGAM run on simulated fMRI data with linear connection functions and pseudo-empirical noise distributions, 200 variables, average degree 10, and sample size 1000. Table 13 shows adjacency precision, adjacency recall, orientation precision, and orientation recall. Table 14 shows the BIC score difference compared to the true graph, number of edges in the estimated graph, and elapsed wall time in seconds. Mean statistics over 20 repetitions are shown with standard deviations in parentheses.

Tables 15 and 16 tabulate results for BOSS and fGES run on the empirical fMRI data described in the main text. Table 15 shows BIC scores, number of undirected edges, and total number of edges. Table 16 shows the improvement in BIC score and change in number of edges when comparing BOSS to fGES. Mean statistics over 171 scans are shown with standard deviations in parentheses.

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Table 1: Erdős-Rényi - Mean (Standard Deviation) - 20 Repetitions

Avg Deg	Algorithm	Adj Pre	Adj Rec	Ori Pre	Ori Rec
2	BOSS	0.96 (0.021)	0.82 (0.033)	0.93 (0.044)	0.68 (0.062)
	GRaSP	0.98 (0.014)	0.81 (0.031)	0.94 (0.047)	0.68 (0.070)
	fGES	0.97 (0.019)	0.82 (0.034)	0.91 (0.046)	0.69 (0.064)
	PC	1.00 (0.005)	0.76 (0.030)	0.66 (0.104)	0.41 (0.068)
	DAGMA	0.99 (0.012)	0.65 (0.047)	0.91 (0.077)	0.34 (0.066)
4	BOSS	0.98 (0.012)	0.82 (0.026)	0.95 (0.031)	0.75 (0.042)
	GRaSP	0.99 (0.005)	0.81 (0.025)	0.95 (0.023)	0.75 (0.041)
	fGES	0.97 (0.019)	0.80 (0.029)	0.88 (0.037)	0.71 (0.047)
	PC	1.00 (0.000)	0.70 (0.031)	0.59 (0.051)	0.37 (0.039)
	DAGMA	0.99 (0.007)	0.61 (0.051)	0.87 (0.034)	0.45 (0.067)
6	BOSS	0.98 (0.013)	0.81 (0.023)	0.96 (0.027)	0.77 (0.036)
	GRaSP	0.99 (0.009)	0.81 (0.021)	0.97 (0.019)	0.77 (0.032)
	fGES	0.95 (0.018)	0.74 (0.024)	0.85 (0.037)	0.65 (0.037)
	PC	1.00 (0.003)	0.64 (0.024)	0.53 (0.043)	0.32 (0.035)
	DAGMA	1.00 (0.005)	0.57 (0.042)	0.89 (0.026)	0.46 (0.048)
8	BOSS	0.98 (0.010)	0.80 (0.022)	0.96 (0.017)	0.77 (0.029)
	GRaSP	0.99 (0.008)	0.80 (0.019)	0.97 (0.014)	0.77 (0.025)
	fGES	0.94 (0.015)	0.65 (0.017)	0.82 (0.050)	0.56 (0.038)
	PC	1.00 (0.003)	0.57 (0.017)	0.52 (0.050)	0.28 (0.027)
	DAGMA	1.00 (0.003)	0.49 (0.066)	0.92 (0.025)	0.42 (0.068)
10	BOSS	0.99 (0.007)	0.81 (0.018)	0.97 (0.013)	0.78 (0.019)
	GRaSP	0.99 (0.007)	0.80 (0.019)	0.97 (0.012)	0.78 (0.019)
	fGES	0.93 (0.017)	0.57 (0.013)	0.80 (0.041)	0.48 (0.029)
	PC	1.00 (0.003)	0.51 (0.015)	0.50 (0.051)	0.24 (0.025)
	DAGMA	1.00 (0.003)	0.41 (0.077)	0.92 (0.024)	0.35 (0.081)
12	BOSS	0.99 (0.008)	0.80 (0.015)	0.97 (0.018)	0.79 (0.020)
	GRaSP	0.99 (0.006)	0.80 (0.013)	0.98 (0.008)	0.79 (0.016)
	fGES	0.93 (0.010)	0.50 (0.007)	0.76 (0.036)	0.41 (0.019)
	PC	0.99 (0.005)	0.45 (0.016)	0.50 (0.040)	0.21 (0.019)
	DAGMA	1.00 (0.002)	0.35 (0.056)	0.91 (0.016)	0.30 (0.055)
14	BOSS	0.99 (0.005)	0.80 (0.016)	0.98 (0.009)	0.78 (0.018)
	GRaSP	0.99 (0.006)	0.80 (0.015)	0.98 (0.012)	0.78 (0.016)
	fGES	0.92 (0.016)	0.43 (0.008)	0.74 (0.041)	0.35 (0.020)
	PC	0.99 (0.004)	0.40 (0.014)	0.51 (0.040)	0.19 (0.017)
	DAGMA	1.00 (0.002)	0.27 (0.053)	0.90 (0.024)	0.23 (0.054)
16	BOSS	0.99 (0.008)	0.79 (0.015)	0.98 (0.014)	0.78 (0.019)
	GRaSP	0.99 (0.012)	0.79 (0.012)	0.98 (0.016)	0.78 (0.014)
	fGES	0.91 (0.018)	0.38 (0.007)	0.71 (0.038)	0.29 (0.017)
	PC	0.98 (0.010)	0.34 (0.013)	0.49 (0.050)	0.16 (0.018)
	DAGMA	1.00 (0.003)	0.23 (0.046)	0.88 (0.029)	0.19 (0.051)
18	BOSS	0.98 (0.007)	0.79 (0.017)	0.97 (0.011)	0.77 (0.020)
	GRaSP	0.98 (0.012)	0.78 (0.017)	0.97 (0.019)	0.77 (0.018)
	fGES	0.90 (0.019)	0.34 (0.007)	0.67 (0.046)	0.25 (0.017)
	PC	0.97 (0.014)	0.31 (0.012)	0.49 (0.042)	0.15 (0.014)
	DAGMA	1.00 (0.004)	0.20 (0.031)	0.87 (0.031)	0.16 (0.032)
20	BOSS	0.98 (0.009)	0.78 (0.017)	0.97 (0.011)	0.77 (0.019)
	GRaSP	0.99 (0.011)	0.77 (0.017)	0.98 (0.018)	0.76 (0.021)
	fGES	0.89 (0.028)	0.30 (0.010)	0.63 (0.041)	0.21 (0.014)
	PC	0.97 (0.012)	0.27 (0.010)	0.47 (0.042)	0.12 (0.012)
	DAGMA	0.99 (0.005)	0.16 (0.021)	0.83 (0.040)	0.12 (0.021)

Table 2: Erdős-Rényi - Mean (Standard Deviation) - 20 Repetitions

Avg Deg	Algorithm	Δ BIC	Edges	Seconds
2	BOSS	-29.27 (12.132)	85.05 (3.886)	0.29 (0.058)
	GRaSP	-26.82 (13.207)	83.05 (3.426)	0.64 (0.268)
	fGES	-29.07 (11.938)	84.35 (3.703)	0.08 (0.018)
	PC	315.45 (125.720)	76.30 (3.164)	0.04 (0.016)
	DAGMA	257.81 (59.894)	66.25 (4.778)	24.43 (8.240)
4	BOSS	-31.87 (20.829)	166.80 (6.110)	0.59 (0.099)
	GRaSP	-32.87 (16.543)	163.55 (5.316)	0.72 (0.121)
	fGES	59.78 (70.153)	166.10 (6.528)	0.12 (0.023)
	PC	1497.21 (324.147)	140.75 (6.290)	0.05 (0.012)
	DAGMA	835.05 (297.781)	123.55 (10.190)	19.97 (6.245)
6	BOSS	-37.73 (22.143)	247.85 (7.734)	1.07 (0.118)
	GRaSP	-46.35 (27.533)	244.75 (6.568)	1.22 (0.168)
	fGES	514.82 (211.383)	234.85 (8.387)	0.19 (0.027)
	PC	3434.62 (486.871)	193.25 (6.866)	0.12 (0.025)
	DAGMA	1593.69 (528.149)	172.80 (12.948)	20.58 (7.643)
8	BOSS	-52.29 (32.968)	327.95 (8.636)	1.90 (0.192)
	GRaSP	-67.02 (28.251)	323.70 (7.020)	2.41 (0.244)
	fGES	1629.07 (333.928)	278.55 (6.117)	0.26 (0.041)
	PC	5456.39 (564.159)	229.35 (6.777)	0.22 (0.036)
	DAGMA	3277.97 (1226.426)	197.85 (26.442)	20.27 (5.811)
10	BOSS	-67.96 (42.305)	408.55 (9.971)	3.23 (0.464)
	GRaSP	-73.79 (40.493)	405.55 (9.784)	4.52 (0.606)
	fGES	3351.28 (525.921)	307.65 (6.643)	0.31 (0.038)
	PC	7970.91 (792.056)	255.60 (7.358)	0.32 (0.039)
	DAGMA	5983.26 (2160.954)	203.70 (38.715)	34.41 (4.913)
12	BOSS	-76.27 (61.569)	488.20 (9.507)	4.99 (0.762)
	GRaSP	-114.59 (31.515)	484.55 (7.964)	7.07 (1.233)
	fGES	5546.54 (523.233)	323.30 (4.281)	0.40 (0.056)
	PC	10445.14 (945.973)	269.55 (9.545)	0.43 (0.056)
	DAGMA	8624.24 (2164.721)	211.65 (33.585)	31.87 (7.555)
14	BOSS	-89.58 (53.599)	566.10 (11.836)	7.48 (0.794)
	GRaSP	-111.92 (49.533)	561.95 (10.450)	10.59 (1.778)
	fGES	7855.75 (678.104)	330.45 (3.706)	0.42 (0.047)
	PC	12864.55 (944.373)	279.45 (10.195)	0.53 (0.059)
	DAGMA	12696.05 (2397.839)	192.25 (37.428)	22.14 (3.549)
16	BOSS	-139.30 (68.319)	643.30 (10.193)	10.37 (0.731)
	GRaSP	-122.77 (90.846)	639.85 (12.554)	15.06 (2.132)
	fGES	10013.10 (701.414)	336.95 (2.373)	0.45 (0.069)
	PC	15568.38 (822.299)	279.30 (9.050)	0.55 (0.064)
	DAGMA	15709.67 (2347.704)	188.10 (36.784)	17.98 (5.559)
18	BOSS	-87.23 (61.003)	719.55 (17.111)	14.08 (1.389)
	GRaSP	-42.03 (116.929)	717.40 (16.282)	19.36 (2.893)
	fGES	12601.28 (911.461)	340.25 (1.803)	0.50 (0.055)
	PC	18390.06 (1218.255)	282.40 (9.162)	0.67 (0.071)
	DAGMA	19095.28 (2239.396)	183.75 (28.703)	19.83 (5.130)
20	BOSS	-112.38 (90.839)	790.05 (15.998)	17.89 (2.214)
	GRaSP	-121.13 (97.982)	786.15 (15.267)	25.49 (3.621)
	fGES	14349.31 (935.139)	342.25 (1.803)	0.53 (0.065)
	PC	20333.60 (1284.439)	279.65 (9.281)	0.72 (0.046)
	DAGMA	23071.47 (2227.819)	156.25 (20.326)	20.42 (4.417)

Table 3: Scale-Free - Mean (Standard Deviation) - 20 Repetitions

Avg Deg	Algorithm	Adj Pre	Adj Rec	Ori Pre	Ori Rec
2	BOSS	0.98 (0.018)	0.82 (0.042)	0.94 (0.044)	0.70 (0.077)
	GRaSP	0.99 (0.014)	0.82 (0.038)	0.95 (0.038)	0.70 (0.075)
	fGES	0.96 (0.034)	0.81 (0.044)	0.90 (0.081)	0.69 (0.089)
	PC	1.00 (0.003)	0.75 (0.039)	0.61 (0.085)	0.39 (0.076)
	DAGMA	0.99 (0.010)	0.67 (0.038)	0.92 (0.042)	0.38 (0.077)
4	BOSS	0.99 (0.009)	0.83 (0.015)	0.97 (0.021)	0.77 (0.036)
	GRaSP	0.99 (0.007)	0.82 (0.016)	0.96 (0.025)	0.77 (0.037)
	fGES	0.86 (0.067)	0.71 (0.047)	0.78 (0.081)	0.63 (0.051)
	PC	1.00 (0.004)	0.67 (0.030)	0.52 (0.072)	0.31 (0.048)
	DAGMA	1.00 (0.009)	0.63 (0.041)	0.92 (0.037)	0.48 (0.057)
6	BOSS	0.99 (0.008)	0.81 (0.026)	0.97 (0.017)	0.77 (0.033)
	GRaSP	0.99 (0.006)	0.80 (0.026)	0.96 (0.020)	0.76 (0.033)
	fGES	0.80 (0.060)	0.59 (0.044)	0.70 (0.072)	0.51 (0.050)
	PC	1.00 (0.004)	0.57 (0.025)	0.50 (0.086)	0.26 (0.049)
	DAGMA	1.00 (0.005)	0.57 (0.041)	0.91 (0.026)	0.48 (0.052)
8	BOSS	0.99 (0.007)	0.81 (0.022)	0.98 (0.013)	0.79 (0.026)
	GRaSP	0.99 (0.007)	0.80 (0.021)	0.97 (0.018)	0.78 (0.029)
	fGES	0.72 (0.051)	0.47 (0.031)	0.60 (0.069)	0.40 (0.043)
	PC	0.99 (0.007)	0.47 (0.023)	0.49 (0.051)	0.22 (0.030)
	DAGMA	1.00 (0.005)	0.48 (0.052)	0.90 (0.029)	0.40 (0.055)
10	BOSS	0.99 (0.007)	0.79 (0.019)	0.98 (0.013)	0.78 (0.020)
	GRaSP	0.99 (0.008)	0.79 (0.019)	0.97 (0.016)	0.77 (0.021)
	fGES	0.70 (0.036)	0.41 (0.019)	0.56 (0.048)	0.32 (0.029)
	PC	0.98 (0.009)	0.41 (0.017)	0.50 (0.057)	0.20 (0.027)
	DAGMA	1.00 (0.002)	0.41 (0.056)	0.90 (0.028)	0.35 (0.059)
12	BOSS	0.99 (0.006)	0.79 (0.021)	0.98 (0.010)	0.78 (0.024)
	GRaSP	0.99 (0.005)	0.79 (0.018)	0.97 (0.013)	0.77 (0.022)
	fGES	0.70 (0.045)	0.36 (0.024)	0.54 (0.047)	0.28 (0.023)
	PC	0.96 (0.015)	0.35 (0.018)	0.46 (0.035)	0.16 (0.017)
	DAGMA	1.00 (0.003)	0.36 (0.050)	0.88 (0.020)	0.31 (0.046)
14	BOSS	0.99 (0.005)	0.79 (0.016)	0.98 (0.008)	0.78 (0.019)
	GRaSP	0.99 (0.006)	0.78 (0.017)	0.97 (0.011)	0.77 (0.020)
	fGES	0.67 (0.029)	0.30 (0.014)	0.49 (0.044)	0.22 (0.020)
	PC	0.94 (0.019)	0.30 (0.012)	0.46 (0.046)	0.14 (0.017)
	DAGMA	0.99 (0.006)	0.31 (0.034)	0.86 (0.036)	0.25 (0.031)
16	BOSS	0.99 (0.005)	0.79 (0.012)	0.98 (0.008)	0.77 (0.012)
	GRaSP	0.98 (0.009)	0.78 (0.011)	0.97 (0.017)	0.76 (0.015)
	fGES	0.67 (0.027)	0.27 (0.011)	0.49 (0.028)	0.20 (0.011)
	PC	0.92 (0.017)	0.27 (0.012)	0.44 (0.065)	0.12 (0.021)
	DAGMA	0.99 (0.006)	0.24 (0.037)	0.86 (0.026)	0.19 (0.036)
18	BOSS	0.98 (0.005)	0.77 (0.014)	0.97 (0.006)	0.76 (0.015)
	GRaSP	0.98 (0.009)	0.77 (0.014)	0.97 (0.013)	0.75 (0.017)
	fGES	0.66 (0.031)	0.24 (0.012)	0.46 (0.038)	0.17 (0.015)
	PC	0.88 (0.023)	0.23 (0.014)	0.46 (0.069)	0.11 (0.019)
	DAGMA	0.99 (0.007)	0.21 (0.043)	0.83 (0.032)	0.16 (0.038)
20	BOSS	0.98 (0.007)	0.76 (0.017)	0.97 (0.010)	0.75 (0.018)
	GRaSP	0.98 (0.008)	0.75 (0.015)	0.97 (0.013)	0.74 (0.016)
	fGES	0.67 (0.037)	0.22 (0.012)	0.45 (0.040)	0.15 (0.014)
	PC	0.87 (0.029)	0.21 (0.014)	0.43 (0.064)	0.10 (0.016)
	DAGMA	0.98 (0.010)	0.18 (0.035)	0.81 (0.028)	0.14 (0.032)

Table 4: Scale-Free - Mean (Standard Deviation) - 20 Repetitions

Avg Deg	Algorithm	Δ BIC	Edges	Seconds
2	BOSS	-26.55 (11.404)	84.00 (4.078)	0.30 (0.073)
	GRaSP	-23.59 (13.066)	82.60 (4.018)	0.50 (0.211)
	fGES	-0.54 (43.742)	83.90 (4.518)	0.07 (0.022)
	PC	470.87 (144.483)	75.40 (3.952)	0.04 (0.050)
	DAGMA	253.53 (47.175)	67.90 (3.932)	29.06 (9.857)
4	BOSS	-43.95 (15.748)	166.85 (2.907)	0.59 (0.088)
	GRaSP	-38.39 (18.982)	165.50 (3.395)	0.87 (0.167)
	fGES	488.50 (242.279)	163.80 (5.634)	0.11 (0.020)
	PC	2082.52 (410.197)	133.55 (6.013)	0.38 (0.348)
	DAGMA	681.29 (213.554)	126.85 (8.400)	26.06 (8.459)
6	BOSS	-72.68 (16.589)	245.20 (8.526)	1.22 (0.210)
	GRaSP	-68.88 (24.561)	243.35 (8.002)	1.78 (0.418)
	fGES	1643.97 (478.571)	221.30 (6.997)	0.17 (0.033)
	PC	4396.89 (778.924)	170.80 (7.090)	0.54 (0.350)
	DAGMA	1509.14 (651.626)	172.95 (12.500)	18.52 (4.705)
8	BOSS	-89.01 (27.280)	327.95 (8.114)	2.22 (0.292)
	GRaSP	-76.04 (31.893)	325.50 (8.217)	3.48 (0.452)
	fGES	3686.06 (482.962)	264.15 (7.073)	0.26 (0.045)
	PC	7147.10 (657.993)	190.05 (8.953)	1.74 (2.189)
	DAGMA	3471.60 (1094.605)	191.45 (20.997)	20.76 (5.768)
10	BOSS	-112.15 (25.346)	401.45 (8.906)	3.65 (0.624)
	GRaSP	-97.90 (29.117)	398.75 (8.271)	6.32 (1.500)
	fGES	5852.32 (740.949)	289.85 (7.856)	0.35 (0.073)
	PC	9700.52 (1032.769)	209.85 (8.216)	1.59 (0.962)
	DAGMA	5733.08 (1669.918)	205.30 (27.891)	21.77 (4.643)
12	BOSS	-133.19 (41.480)	481.30 (11.282)	5.34 (0.550)
	GRaSP	-112.36 (36.297)	479.60 (10.002)	11.10 (3.024)
	fGES	7773.11 (700.188)	306.45 (6.219)	0.40 (0.058)
	PC	12356.57 (757.713)	220.85 (9.275)	1.34 (1.007)
	DAGMA	7951.55 (1554.948)	218.40 (30.797)	23.48 (8.366)
14	BOSS	-153.80 (37.574)	558.30 (11.770)	7.87 (0.954)
	GRaSP	-133.53 (38.246)	555.15 (12.175)	14.85 (2.982)
	fGES	10081.52 (734.713)	316.35 (5.678)	0.47 (0.056)
	PC	15099.54 (983.888)	225.05 (7.244)	1.16 (0.578)
	DAGMA	10753.51 (1787.888)	217.10 (24.531)	35.74 (0.586)
16	BOSS	-158.52 (33.706)	637.15 (9.686)	11.14 (1.657)
	GRaSP	-115.00 (52.270)	633.15 (9.213)	22.15 (3.366)
	fGES	12126.92 (687.750)	325.35 (3.233)	0.51 (0.074)
	PC	17749.99 (1074.091)	233.45 (8.593)	0.92 (0.142)
	DAGMA	15388.97 (2237.310)	191.25 (29.958)	28.44 (8.964)
18	BOSS	-199.97 (33.289)	706.10 (13.286)	14.15 (1.858)
	GRaSP	-157.95 (44.206)	702.20 (12.968)	28.48 (5.480)
	fGES	13650.12 (878.170)	328.10 (5.647)	0.60 (0.073)
	PC	19560.16 (1165.357)	231.35 (9.115)	0.92 (0.155)
	DAGMA	17594.42 (2767.296)	189.85 (39.175)	30.24 (2.855)
20	BOSS	-232.22 (43.870)	771.90 (14.418)	17.12 (1.910)
	GRaSP	-191.62 (47.754)	767.60 (13.228)	33.99 (9.456)
	fGES	15010.88 (1165.819)	332.40 (3.817)	0.57 (0.076)
	PC	21362.04 (1513.351)	238.05 (10.630)	0.95 (0.191)
	DAGMA	19822.75 (3327.041)	186.15 (35.397)	23.74 (5.430)

Table 5: Gumbel - Mean (Standard Deviation) - 20 Repetitions

Avg Deg	Algorithm	Adj Pre	Adj Rec	Ori Pre	Ori Rec
2	BOSS	0.98 (0.021)	0.77 (0.048)	0.91 (0.040)	0.58 (0.084)
	DAGMA	1.00 (0.004)	0.58 (0.047)	0.82 (0.061)	0.47 (0.058)
	LiNGAM	0.86 (0.040)	0.80 (0.045)	0.80 (0.059)	0.78 (0.050)
4	BOSS	0.99 (0.008)	0.75 (0.029)	0.93 (0.029)	0.67 (0.039)
	DAGMA	1.00 (0.002)	0.54 (0.026)	0.87 (0.026)	0.46 (0.030)
	LiNGAM	0.89 (0.027)	0.80 (0.029)	0.86 (0.029)	0.78 (0.031)
6	BOSS	0.99 (0.008)	0.76 (0.025)	0.95 (0.019)	0.71 (0.032)
	DAGMA	1.00 (0.000)	0.50 (0.050)	0.89 (0.026)	0.45 (0.046)
	LiNGAM	0.90 (0.023)	0.81 (0.024)	0.88 (0.030)	0.80 (0.026)
8	BOSS	0.99 (0.007)	0.77 (0.019)	0.96 (0.013)	0.73 (0.026)
	DAGMA	1.00 (0.002)	0.46 (0.050)	0.89 (0.027)	0.41 (0.048)
	LiNGAM	0.91 (0.022)	0.81 (0.017)	0.89 (0.026)	0.80 (0.020)
10	BOSS	0.99 (0.008)	0.76 (0.021)	0.96 (0.013)	0.73 (0.021)
	DAGMA	1.00 (0.001)	0.39 (0.052)	0.90 (0.018)	0.35 (0.047)
	LiNGAM	0.90 (0.021)	0.81 (0.023)	0.89 (0.023)	0.80 (0.024)
12	BOSS	0.99 (0.006)	0.75 (0.017)	0.96 (0.012)	0.72 (0.023)
	DAGMA	1.00 (0.001)	0.32 (0.049)	0.89 (0.016)	0.29 (0.043)
	LiNGAM	0.90 (0.014)	0.81 (0.015)	0.88 (0.019)	0.80 (0.017)
14	BOSS	0.99 (0.005)	0.74 (0.019)	0.97 (0.011)	0.72 (0.020)
	DAGMA	1.00 (0.004)	0.28 (0.032)	0.87 (0.032)	0.25 (0.029)
	LiNGAM	0.90 (0.013)	0.81 (0.018)	0.89 (0.012)	0.80 (0.019)
16	BOSS	0.99 (0.005)	0.72 (0.017)	0.97 (0.013)	0.70 (0.021)
	DAGMA	1.00 (0.003)	0.24 (0.038)	0.86 (0.019)	0.20 (0.034)
	LiNGAM	0.90 (0.015)	0.80 (0.014)	0.89 (0.017)	0.79 (0.015)
18	BOSS	0.99 (0.004)	0.73 (0.016)	0.97 (0.010)	0.71 (0.019)
	DAGMA	1.00 (0.004)	0.21 (0.032)	0.85 (0.028)	0.18 (0.031)
	LiNGAM	0.90 (0.014)	0.81 (0.015)	0.89 (0.016)	0.80 (0.016)
20	BOSS	0.99 (0.004)	0.72 (0.020)	0.97 (0.009)	0.70 (0.023)
	DAGMA	0.99 (0.006)	0.20 (0.035)	0.83 (0.033)	0.17 (0.030)
	LiNGAM	0.91 (0.010)	0.80 (0.014)	0.90 (0.011)	0.80 (0.015)

Table 6: Gumbel - Mean (Standard Deviation) - 20 Repetitions

Avg Deg	Algorithm	Δ BIC	Edges	Seconds
2	BOSS	-40.48 (14.664)	78.55 (5.772)	0.30 (0.058)
	DAGMA	204.95 (76.866)	58.20 (4.786)	15.55 (6.018)
	LiNGAM	-54.83 (13.206)	93.35 (7.415)	105.82 (1.532)
4	BOSS	-65.33 (16.983)	152.70 (5.741)	0.51 (0.073)
	DAGMA	565.94 (126.533)	107.15 (5.153)	13.78 (3.069)
	LiNGAM	-88.19 (18.882)	179.85 (8.412)	104.98 (1.320)
6	BOSS	-90.41 (24.002)	231.80 (7.388)	0.97 (0.141)
	DAGMA	1229.57 (459.846)	151.35 (15.034)	16.07 (3.621)
	LiNGAM	-103.33 (28.705)	271.40 (12.588)	105.10 (0.483)
8	BOSS	-120.74 (24.131)	310.15 (7.659)	1.64 (0.167)
	DAGMA	2291.71 (631.910)	182.55 (19.888)	16.97 (5.174)
	LiNGAM	-128.92 (26.895)	357.05 (11.381)	105.24 (0.611)
10	BOSS	-139.30 (35.203)	382.95 (11.237)	2.69 (0.443)
	DAGMA	3743.50 (1029.405)	194.05 (26.205)	15.91 (5.753)
	LiNGAM	-169.05 (42.471)	450.35 (17.187)	105.66 (0.917)
12	BOSS	-166.78 (25.581)	453.00 (10.110)	4.01 (0.667)
	DAGMA	5723.14 (1473.488)	194.80 (29.269)	17.11 (3.534)
	LiNGAM	-197.03 (35.866)	542.25 (15.297)	106.29 (1.778)
14	BOSS	-196.92 (34.669)	521.00 (13.792)	5.50 (0.634)
	DAGMA	7252.63 (1027.255)	199.95 (22.795)	16.60 (3.512)
	LiNGAM	-236.48 (36.626)	625.10 (16.905)	110.73 (2.599)
16	BOSS	-238.72 (30.030)	585.60 (13.141)	6.93 (0.909)
	DAGMA	9404.55 (1693.592)	190.10 (30.805)	18.25 (5.813)
	LiNGAM	-271.78 (37.077)	711.60 (16.643)	107.66 (3.950)
18	BOSS	-248.95 (44.181)	664.15 (14.496)	9.86 (1.305)
	DAGMA	11752.12 (1845.939)	191.40 (28.914)	37.50 (31.535)
	LiNGAM	-285.60 (49.074)	806.15 (21.475)	105.50 (6.056)
20	BOSS	-278.55 (50.250)	727.00 (19.960)	12.69 (1.851)
	DAGMA	13095.63 (2319.085)	199.75 (35.909)	22.59 (10.840)
	LiNGAM	-334.91 (44.193)	887.20 (20.281)	103.66 (3.493)

Table 7: Exponential - Mean (Standard Deviation) - 20 Repetitions

Avg Deg	Algorithm	Adj Pre	Adj Rec	Ori Pre	Ori Rec
2	BOSS	0.97 (0.016)	0.84 (0.036)	0.94 (0.030)	0.71 (0.075)
	DAGMA	0.99 (0.010)	0.68 (0.050)	0.85 (0.040)	0.58 (0.051)
	LiNGAM	0.88 (0.033)	0.87 (0.034)	0.84 (0.041)	0.87 (0.039)
4	BOSS	0.98 (0.014)	0.81 (0.027)	0.95 (0.018)	0.75 (0.038)
	DAGMA	1.00 (0.005)	0.62 (0.033)	0.88 (0.026)	0.55 (0.036)
	LiNGAM	0.89 (0.030)	0.84 (0.023)	0.88 (0.031)	0.85 (0.023)
6	BOSS	0.99 (0.008)	0.80 (0.020)	0.97 (0.014)	0.77 (0.024)
	DAGMA	1.00 (0.004)	0.56 (0.046)	0.90 (0.029)	0.51 (0.046)
	LiNGAM	0.91 (0.020)	0.84 (0.020)	0.90 (0.021)	0.84 (0.019)
8	BOSS	0.98 (0.010)	0.81 (0.021)	0.97 (0.015)	0.79 (0.025)
	DAGMA	1.00 (0.004)	0.49 (0.047)	0.90 (0.024)	0.44 (0.042)
	LiNGAM	0.91 (0.021)	0.85 (0.018)	0.90 (0.021)	0.85 (0.019)
10	BOSS	0.99 (0.007)	0.80 (0.014)	0.97 (0.013)	0.78 (0.018)
	DAGMA	1.00 (0.003)	0.42 (0.055)	0.90 (0.027)	0.38 (0.051)
	LiNGAM	0.91 (0.017)	0.85 (0.015)	0.91 (0.016)	0.84 (0.015)
12	BOSS	0.99 (0.006)	0.79 (0.017)	0.98 (0.006)	0.77 (0.016)
	DAGMA	1.00 (0.003)	0.36 (0.046)	0.89 (0.022)	0.32 (0.041)
	LiNGAM	0.91 (0.016)	0.84 (0.015)	0.91 (0.016)	0.84 (0.014)
14	BOSS	0.98 (0.006)	0.78 (0.020)	0.98 (0.009)	0.77 (0.021)
	DAGMA	0.99 (0.007)	0.28 (0.036)	0.86 (0.030)	0.24 (0.031)
	LiNGAM	0.90 (0.017)	0.84 (0.015)	0.90 (0.017)	0.84 (0.015)
16	BOSS	0.98 (0.007)	0.77 (0.020)	0.97 (0.011)	0.76 (0.021)
	DAGMA	0.99 (0.006)	0.25 (0.041)	0.84 (0.029)	0.21 (0.039)
	LiNGAM	0.90 (0.012)	0.84 (0.011)	0.90 (0.012)	0.83 (0.012)
18	BOSS	0.98 (0.007)	0.77 (0.012)	0.97 (0.010)	0.76 (0.015)
	DAGMA	0.99 (0.010)	0.20 (0.037)	0.83 (0.036)	0.17 (0.031)
	LiNGAM	0.90 (0.012)	0.84 (0.007)	0.90 (0.013)	0.83 (0.007)
20	BOSS	0.98 (0.004)	0.76 (0.013)	0.97 (0.006)	0.75 (0.014)
	DAGMA	0.98 (0.012)	0.17 (0.043)	0.80 (0.036)	0.14 (0.035)
	LiNGAM	0.91 (0.010)	0.83 (0.009)	0.90 (0.010)	0.83 (0.009)

Table 8: Exponential - Mean (Standard Deviation) - 20 Repetitions

Avg Deg	Algorithm	Δ BIC	Edges	Seconds
2	BOSS	-30.13 (13.273)	86.40 (3.831)	0.34 (0.058)
	DAGMA	242.69 (83.387)	68.85 (4.902)	17.39 (4.199)
	LiNGAM	-44.29 (11.441)	98.65 (5.234)	103.83 (0.493)
4	BOSS	-53.80 (17.446)	165.40 (5.041)	0.62 (0.122)
	DAGMA	671.26 (151.117)	124.85 (6.854)	15.10 (4.342)
	LiNGAM	-81.25 (19.414)	189.25 (6.455)	105.02 (0.398)
6	BOSS	-74.63 (19.124)	245.00 (6.728)	1.28 (0.165)
	DAGMA	1554.85 (463.944)	169.05 (13.904)	15.83 (3.214)
	LiNGAM	-103.35 (13.446)	277.60 (9.029)	105.03 (0.618)
8	BOSS	-93.93 (26.702)	329.55 (8.556)	2.19 (0.258)
	DAGMA	3077.68 (928.322)	198.10 (19.123)	17.80 (3.796)
	LiNGAM	-121.79 (25.327)	374.35 (11.463)	105.08 (0.397)
10	BOSS	-126.28 (26.061)	405.45 (7.287)	3.88 (0.483)
	DAGMA	5243.46 (1178.323)	211.15 (27.910)	17.47 (5.457)
	LiNGAM	-153.33 (25.696)	462.55 (11.569)	105.54 (1.044)
12	BOSS	-140.74 (31.044)	479.25 (9.808)	5.78 (0.864)
	DAGMA	7766.05 (1751.380)	219.50 (28.065)	18.54 (5.163)
	LiNGAM	-172.74 (27.491)	552.15 (9.724)	107.57 (1.642)
14	BOSS	-167.21 (33.004)	557.30 (13.495)	8.53 (1.297)
	DAGMA	11865.65 (2010.847)	197.95 (24.373)	22.40 (8.998)
	LiNGAM	-203.28 (39.098)	651.90 (16.814)	109.91 (4.678)
16	BOSS	-167.46 (38.958)	628.70 (14.120)	11.39 (1.579)
	DAGMA	14316.68 (2391.848)	202.60 (33.271)	19.48 (6.907)
	LiNGAM	-212.26 (30.366)	739.25 (11.543)	109.25 (3.127)
18	BOSS	-206.37 (30.259)	702.35 (10.194)	15.31 (1.998)
	DAGMA	17855.62 (2605.919)	184.10 (33.158)	47.89 (30.300)
	LiNGAM	-249.31 (28.421)	833.00 (15.731)	111.06 (5.913)
20	BOSS	-221.56 (43.075)	770.45 (13.221)	19.62 (2.918)
	DAGMA	21255.44 (3529.781)	171.10 (43.489)	23.55 (5.630)
	LiNGAM	-278.01 (31.714)	921.60 (11.523)	104.27 (0.655)

Table 9: High Dimensional - Mean (Standard Deviation) - 10 Repetitions

Variables	Algorithm	BIC λ	Adj Pre	Adj Rec	Ori Pre	Ori Rec
100	BOSS	2	0.98 (0.005)	0.75 (0.008)	0.97 (0.008)	0.74 (0.009)
	GRaSP		0.98 (0.004)	0.74 (0.009)	0.97 (0.010)	0.73 (0.008)
	BOSS	4	0.99 (0.003)	0.64 (0.014)	0.97 (0.008)	0.62 (0.014)
	GRaSP		0.99 (0.006)	0.63 (0.016)	0.96 (0.013)	0.61 (0.018)
	BOSS	8	0.98 (0.004)	0.47 (0.009)	0.94 (0.014)	0.44 (0.014)
	GRaSP		0.97 (0.007)	0.45 (0.014)	0.90 (0.023)	0.41 (0.020)
200	BOSS	2	0.98 (0.004)	0.78 (0.013)	0.98 (0.005)	0.77 (0.013)
	GRaSP		0.98 (0.004)	0.78 (0.013)	0.98 (0.005)	0.77 (0.012)
	BOSS	4	0.99 (0.002)	0.69 (0.015)	0.99 (0.005)	0.68 (0.015)
	GRaSP		0.99 (0.003)	0.68 (0.015)	0.98 (0.006)	0.67 (0.018)
	BOSS	8	0.99 (0.004)	0.53 (0.015)	0.97 (0.004)	0.52 (0.016)
	GRaSP		0.99 (0.004)	0.52 (0.017)	0.96 (0.007)	0.51 (0.020)
300	BOSS	2	0.98 (0.003)	0.79 (0.006)	0.98 (0.003)	0.79 (0.007)
	GRaSP		0.99 (0.003)	0.79 (0.007)	0.98 (0.004)	0.79 (0.008)
	BOSS	4	0.99 (0.002)	0.71 (0.009)	0.99 (0.002)	0.70 (0.009)
	GRaSP		1.00 (0.002)	0.71 (0.009)	0.99 (0.002)	0.70 (0.009)
	BOSS	8	0.99 (0.003)	0.55 (0.011)	0.98 (0.005)	0.54 (0.012)
	GRaSP		0.99 (0.003)	0.55 (0.012)	0.98 (0.007)	0.54 (0.015)
400	BOSS	2	0.98 (0.003)	0.79 (0.005)	0.97 (0.003)	0.79 (0.005)
	GRaSP		0.98 (0.003)	0.79 (0.004)	0.98 (0.003)	0.79 (0.004)
	BOSS	4	1.00 (0.002)	0.71 (0.008)	0.99 (0.003)	0.70 (0.009)
	GRaSP		1.00 (0.001)	0.71 (0.008)	0.99 (0.003)	0.70 (0.009)
	BOSS	8	0.99 (0.002)	0.56 (0.012)	0.99 (0.003)	0.55 (0.014)
	GRaSP		0.99 (0.002)	0.55 (0.015)	0.98 (0.004)	0.54 (0.016)
500	BOSS	2	0.98 (0.003)	0.80 (0.005)	0.97 (0.003)	0.80 (0.005)
	GRaSP		0.98 (0.003)	0.80 (0.005)	0.98 (0.003)	0.80 (0.005)
	BOSS	4	1.00 (0.001)	0.72 (0.007)	0.99 (0.002)	0.72 (0.007)
	GRaSP		1.00 (0.001)	0.72 (0.007)	0.99 (0.002)	0.72 (0.008)
	BOSS	8	1.00 (0.002)	0.57 (0.008)	0.99 (0.003)	0.57 (0.008)
	GRaSP		1.00 (0.002)	0.57 (0.009)	0.99 (0.004)	0.56 (0.009)
600	BOSS	2	0.98 (0.002)	0.80 (0.005)	0.97 (0.002)	0.80 (0.005)
	GRaSP		0.98 (0.002)	0.80 (0.005)	0.98 (0.002)	0.80 (0.005)
	BOSS	4	1.00 (0.001)	0.72 (0.004)	0.99 (0.001)	0.72 (0.004)
	GRaSP		1.00 (0.001)	0.72 (0.005)	0.99 (0.001)	0.71 (0.005)
	BOSS	8	1.00 (0.001)	0.57 (0.009)	0.99 (0.002)	0.57 (0.009)
	GRaSP		1.00 (0.001)	0.57 (0.009)	0.99 (0.002)	0.56 (0.008)

Table 10: High Dimensional - Mean (Standard Deviation) - 10 Repetitions

Variables	Algorithm	BIC λ	Adj Pre	Adj Rec	Ori Pre	Ori Rec
700	BOSS	2	0.97 (0.003)	0.80 (0.006)	0.97 (0.003)	0.80 (0.006)
	GRaSP		0.98 (0.002)	0.80 (0.006)	0.97 (0.003)	0.80 (0.006)
	BOSS	4	1.00 (0.001)	0.72 (0.005)	0.99 (0.001)	0.72 (0.005)
	GRaSP		1.00 (0.001)	0.72 (0.005)	0.99 (0.001)	0.72 (0.005)
	BOSS	8	1.00 (0.001)	0.58 (0.006)	0.99 (0.002)	0.58 (0.006)
	GRaSP		1.00 (0.001)	0.58 (0.007)	0.99 (0.002)	0.57 (0.007)
800	BOSS	2	0.97 (0.002)	0.80 (0.005)	0.97 (0.003)	0.80 (0.005)
	GRaSP		1.00 (0.001)	0.72 (0.005)	0.99 (0.001)	0.72 (0.005)
	BOSS	4	1.00 (0.001)	0.72 (0.006)	0.99 (0.001)	0.72 (0.005)
	GRaSP		1.00 (0.001)	0.72 (0.006)	0.99 (0.002)	0.58 (0.005)
	BOSS	8	1.00 (0.001)	0.58 (0.005)	0.99 (0.002)	0.58 (0.005)
	GRaSP		1.00 (0.001)	0.58 (0.006)	0.99 (0.002)	0.57 (0.006)
900	BOSS	2	0.97 (0.002)	0.80 (0.004)	0.97 (0.002)	0.80 (0.004)
	GRaSP		1.00 (0.001)	0.73 (0.004)	1.00 (0.001)	0.72 (0.004)
	BOSS	4	1.00 (0.000)	0.73 (0.004)	1.00 (0.001)	0.72 (0.004)
	GRaSP		1.00 (0.001)	0.59 (0.006)	1.00 (0.001)	0.58 (0.006)
	BOSS	8	1.00 (0.001)	0.58 (0.006)	0.99 (0.002)	0.58 (0.006)
	GRaSP		1.00 (0.001)	0.58 (0.006)	0.99 (0.002)	0.58 (0.006)
1000	BOSS	2	0.97 (0.002)	0.80 (0.003)	0.97 (0.002)	0.8 (0.003)
	GRaSP		1.00 (0.001)	0.73 (0.006)	1.0 (0.002)	0.72 (0.006)
	BOSS	4	1.00 (0.001)	0.72 (0.006)	1.00 (0.001)	0.72 (0.006)
	GRaSP		1.00 (0.001)	0.59 (0.006)	1.00 (0.001)	0.58 (0.006)
	BOSS	8	1.00 (0.001)	0.59 (0.006)	1.00 (0.001)	0.58 (0.006)
	GRaSP		1.00 (0.001)	0.59 (0.006)	1.00 (0.001)	0.58 (0.006)

Table 11: High Dimensional - Mean (Standard Deviation) - 10 Repetitions

Variables	Algorithm	BIC λ	Δ BIC	Edges	Seconds
100	BOSS	2	-250.40 (37.438)	762.7 (6.533)	16.88 (2.089)
	GRaSP		-219.89 (38.486)	756.4 (9.419)	29.21 (3.756)
	BOSS	4	680.01 (115.182)	646.3 (14.568)	5.16 (0.747)
	GRaSP		748.13 (121.240)	642.3 (14.659)	9.94 (1.854)
	BOSS	8	4065.45 (294.262)	477.6 (9.857)	1.74 (0.255)
	GRaSP		4826.93 (546.131)	458.4 (13.032)	2.61 (0.434)
200	BOSS	2	-449.04 (50.265)	1587.0 (20.827)	110.62 (11.576)
	GRaSP		-401.91 (81.173)	1578.2 (22.115)	226.31 (17.332)
	BOSS	4	1062.78 (139.848)	1389.9 (28.938)	27.41 (3.499)
	GRaSP		1234.52 (171.907)	1377.0 (29.405)	59.14 (6.809)
	BOSS	8	7386.30 (418.935)	1069.0 (32.156)	8.74 (0.559)
	GRaSP		7943.37 (357.622)	1054.8 (35.333)	16.02 (2.520)
300	BOSS	2	-697.19 (58.726)	2428.8 (15.852)	320.69 (24.340)
	GRaSP		-650.07 (44.280)	2417.1 (20.091)	689.65 (77.067)
	BOSS	4	1391.78 (159.564)	2141.4 (24.613)	71.44 (6.100)
	GRaSP		1487.97 (163.380)	2133.7 (25.578)	155.94 (15.237)
	BOSS	8	10668.10 (640.734)	1666.2 (32.519)	21.79 (1.769)
	GRaSP		11103.41 (693.257)	1651.4 (35.034)	59.82 (10.100)
400	BOSS	2	-989.29 (116.766)	3245.7 (15.384)	584.74 (57.027)
	GRaSP		-945.61 (9.502)	3231.2 (17.869)	382.63 (57.505)
	BOSS	4	1946.53 (303.747)	2843.5 (30.642)	133.17 (9.047)
	GRaSP		2029.54 (257.483)	2834.7 (31.489)	330.64 (19.780)
	BOSS	8	13366.98 (868.069)	2245.6 (50.000)	43.28 (3.978)
	GRaSP		14230.16 (1053.206)	2218.4 (57.618)	114.04 (11.378)
500	BOSS	2	-1282.97 (48.490)	4102.1 (26.860)	1012.22 (60.835)
	GRaSP		-1239.31 (36.450)	4085.2 (29.001)	2331.55 (249.112)
	BOSS	4	2168.07 (253.685)	3619.0 (34.438)	214.01 (9.017)
	GRaSP		2202.18 (277.513)	3609.7 (36.794)	637.89 (61.230)
	BOSS	8	16143.34 (783.084)	2880.1 (37.513)	77.40 (7.527)
	GRaSP		16946.30 (1096.986)	2856.2 (40.386)	233.62 (23.181)
600	BOSS	2	-1573.63 (118.084)	4924.2 (28.358)	1483.88 (97.721)
	GRaSP		-1542.92 (108.987)	4909.1 (30.443)	3755.78 (677.803)
	BOSS	4	2599.43 (184.187)	4332.9 (29.027)	320.71 (22.809)
	GRaSP		2713.14 (166.374)	4319.4 (28.776)	1018.19 (101.612)
	BOSS	8	18967.19 (779.458)	3450.9 (53.978)	117.72 (12.370)
	GRaSP		19657.44 (849.114)	3426.7 (55.140)	427.71 (42.478)

Table 12: High Dimensional - Mean (Standard Deviation) - 10 Repetitions

Variables	Algorithm	BIC λ	Δ BIC	Edges	Seconds
700	BOSS	2	-1899.57 (134.364)	5773.7 (33.036)	2453.32 (100.902)
	GRaSP		-1865.52 (120.199)	5749.2 (37.611)	5758.49 (564.612)
	BOSS	4	2995.26 (240.846)	5067.6 (39.973)	456.97 (28.944)
	GRaSP		3023.06 (225.375)	5056.4 (38.208)	1430.72 (106.472)
	BOSS	8	21327.81 (719.142)	4086.1 (43.144)	178.79 (11.971)
	GRaSP		21783.71 (890.414)	4067.9 (44.851)	698.20 (71.643)
800	BOSS	2	-2210.01 (87.533)	6617.1 (37.072)	3083.87 (225.958)
	GRaSP				
	BOSS	4	3395.12 (215.775)	5795.3 (42.008)	702.54 (55.730)
	GRaSP		3547.25 (256.39)	5777.7 (46.159)	2103.76 (167.556)
	BOSS	8	24391.82 (886.115)	4668.5 (40.768)	282.38 (23.453)
	GRaSP		24820.67 (708.219)	4650.9 (42.956)	1052.97 (87.079)
900	BOSS	2	-2579.65 (113.671)	7470.1 (32.306)	4051.92 (113.881)
	GRaSP				
	BOSS	4	3615.29 (226.659)	6556.4 (30.310)	880.03 (47.222)
	GRaSP		3763.39 (257.163)	6539.1 (31.125)	2832.26 (196.926)
	BOSS	8	27255.29 (865.222)	5284.7 (50.352)	471.74 (43.150)
	GRaSP		27741.60 (978.542)	5262.7 (55.201)	1508.55 (162.988)
1000	BOSS	2	-2921.0 (153.220)	8315.8 (22.220)	5113.12 (440.444)
	GRaSP				
	BOSS	4	4071.33 (371.126)	7274.1 (54.519)	1113.69 (61.362)
	GRaSP		4246.67 (452.957)	7257.4 (57.57)	3730.44 (375.983)
	BOSS	8	29520.77 (1101.301)	5889.5 (57.853)	599.24 (43.804)
	GRaSP		30005.90 (1233.898)	5870.1 (60.090)	2176.70 (203.593)

Table 13: fMRI Simulation - Mean (Standard Deviation) - 40 Repetitions

Algorithm	Adj Pre	Adj Rec	Ori Pre	Ori Rec
BOSS	0.99 (0.005)	0.94 (0.009)	0.96 (0.009)	0.90 (0.012)
DAGMA	1.00 (0.000)	0.69 (0.038)	0.98 (0.005)	0.67 (0.037)
fGES	0.97 (0.007)	0.60 (0.012)	0.70 (0.038)	0.43 (0.025)
LiNGAM	0.54 (0.026)	0.94 (0.008)	0.35 (0.031)	0.62 (0.032)

Table 14: fMRI Simulation - Mean (Standard Deviation) - 40 Repetitions

Algorithm	Δ BIC	Edges	Seconds
BOSS	211.79 (85.026)	951.40 (23.546)	15.46 (1.987)
DAGMA	3080.85 (853.227)	687.15 (40.417)	54.58 (18.957)
fGES	7784.48 (751.306)	617.35 (7.033)	5.12 (1.344)
LiNGAM	3868.93 (376.249)	1752.75 (99.268)	582.03 (3.278)

Table 15: fMRI - Mean (Standard Deviation) - 171 Scans

Algorithm	BIC	Undirected Edges	Total Edges
BOSS	98752.23 (25139.907)	3.09 (2.189)	2644.61 (430.419)
fGES	96171.56 (24592.247)	5.79 (7.083)	2452.94 (394.0745)

Table 16: fMRI - Mean (Standard Deviation) - 171 Scans

Δ Algorithm	Δ BIC	Δ Edges
BOSS - fGES	2580.68 (664.574)	191.67 (55.961)