Supplementary Material

A Tsallis-perspective: Proofs

To prove Lemma 1 we need the following technical result that gives an expression for the Hessian of the Tsallis-perspective H in terms of the (scalar) derivatives of h.

Lemma 6. *The Hessian of H (Eq. (1)) at any point $x \in \mathbb{R}_+^d$ can be expressed as:*

$$\begin{aligned}\nabla^2 H(x) = &-\frac{1}{4}\|x\|_1^{-\frac{3}{2}} \sum_{i=1}^{d} h\Big(\frac{x_i}{\|x\|_1}\Big) z z^\top \\ &+ \|x\|_1^{-\frac{7}{2}} \sum_{i=1}^{d} x_i^2 h''\Big(\frac{x_i}{\|x\|_1}\Big) z_i z_i^\top \\ &+ \frac{1}{2}\|x\|_1^{-\frac{5}{2}} \sum_{i=1}^{d} x_i h'\Big(\frac{x_i}{\|x\|_1}\Big)\big(z z_i^\top + z_i z^\top\big) ,\end{aligned}$$

where $z = \mathbf{1}_d$ is the all-ones vector, and $z_i = \mathbf{1}_d - (\|x\|_1/x_i)\mathbf{e}_i$ for all $i \in [d]$.

Proof. Let us first compute the first and second derivatives of $f(x) = \sqrt{\|x\|_1}$ and $g_i(x) = h(x_i/\|x\|_1)$ for a fixed $i \in [d]$:

$$\begin{aligned}\nabla f(x) &= \frac{1}{2}\|x\|_1^{-\frac{1}{2}} z; \\ \nabla^2 f(x) &= -\frac{1}{4}\|x\|_1^{-\frac{3}{2}} z z^\top; \\ \nabla g_i(x) &= h'\Big(\frac{x_i}{\|x\|_1}\Big)\Big(\frac{1}{\|x\|_1}\mathbf{e}_i - \frac{x_i}{\|x\|_1^2} z\Big) \\ &= -\frac{x_i}{\|x\|_1^2} h'\Big(\frac{x_i}{\|x\|_1}\Big) z_i; \\ \nabla^2 g_i(x) &= h''\Big(\frac{x_i}{\|x\|_1}\Big)\Big(\frac{1}{\|x\|_1}\mathbf{e}_i - \frac{x_i}{\|x\|_1^2} z\Big)\Big(\frac{1}{\|x\|_1}\mathbf{e}_i - \frac{x_i}{\|x\|_1^2} z\Big)^\top \\ &\quad + h'\Big(\frac{x_i}{\|x\|_1}\Big)\Big(-\frac{1}{\|x\|_1^2} z \mathbf{e}_i^\top - \frac{1}{\|x\|_1^2}\mathbf{e}_i z^\top + \frac{2x_i}{\|x\|_1^3} z z^\top\Big) \\ &= \frac{x_i^2}{\|x\|_1^4} h''\Big(\frac{x_i}{\|x\|_1}\Big) z_i z_i^\top + \frac{x_i}{\|x\|_1^3} h'\Big(\frac{x_i}{\|x\|_1}\Big)(z z_i^\top + z_i z^\top).\end{aligned}$$

Using the formula for the Hessian of a product, we now obtain:

$$\begin{aligned}&\nabla^2\big(f(x) g_i(x)\big) \\ &\quad = \big(\nabla^2 f(x)\big) g_i(x) + \nabla f(x) \nabla g_i(x)^\top + \nabla g_i(x) \nabla f(x)^\top + f(x)\big(\nabla^2 g_i(x)\big) \\ &\quad = -\frac{1}{4}\|x\|_1^{-\frac{3}{2}} h\Big(\frac{x_i}{\|x\|_1}\Big) z z^\top - \frac{1}{2}\|x\|_1^{-\frac{5}{2}} x_i h'\Big(\frac{x_i}{\|x\|_1}\Big)\big(z z_i^\top + z_i z^\top\big) \\ &\quad\ \ + \|x\|_1^{-\frac{7}{2}} x_i^2 h''\Big(\frac{x_i}{\|x\|_1}\Big) z_i z_i^\top + \|x\|_1^{-\frac{5}{2}} x_i h'\Big(\frac{x_i}{\|x\|_1}\Big)\big(z z_i^\top + z_i z^\top\big) \\ &\quad = -\frac{1}{4}\|x\|_1^{-\frac{3}{2}} h\Big(\frac{x_i}{\|x\|_1}\Big) z z^\top + \|x\|_1^{-\frac{7}{2}} x_i^2 h''\Big(\frac{x_i}{\|x\|_1}\Big) z_i z_i^\top + \frac{1}{2}\|x\|_1^{-\frac{5}{2}} x_i h'\Big(\frac{x_i}{\|x\|_1}\Big)\big(z z_i^\top + z_i z^\top\big).\end{aligned}$$

Summing this over $i = 1, \ldots, d$, we obtain the expression for the Hessian $\nabla^2 H(x)$. ■

Proof of Lemma 1. Fix $x \in \mathbb{R}_+^d$ and let $y_i = x_i/\|x\|_1$ for all i. By Lemma 6 the Hessian of H can be written as

$$\nabla^2 H(x) = \frac{1}{4}\|x\|_1^{-\frac{3}{2}} \sum_{i=1}^{d} \Big(-h(y_i)zz^\top + 4y_i^2 h''(y_i) z_i z_i^\top + 2y_i h'(y_i)\big(zz_i^\top + z_i z^\top\big)\Big),$$

where $z = \mathbf{1}_d$ is the all-ones vector, and $z_i = \mathbf{1}_d - (\|x\|_1/x_i)\mathbf{e}_i$ for all $i \in [d]$. Then, using the condition on h and since $\sum_{i=1}^d y_i z_i = 0$ we have

$$\begin{aligned}
\nabla^2 H(x) &\geq \frac{\lambda_h}{4}\|x\|_1^{-\frac{3}{2}} J + \frac{1}{4}\|x\|_1^{-\frac{3}{2}} \sum_{i=1}^{d}\left(\frac{\big(h'(y_i) - c_h\big)^2}{\frac{1}{2}h''(y_i)} zz^\top + 4y_i^2 h''(y_i) z_i z_i^\top + 2y_i h'(y_i)\big(zz_i^\top + z_i z^\top\big)\right) \\
&= \frac{\lambda_h}{4}\|x\|_1^{-\frac{3}{2}} J + \frac{1}{4}\|x\|_1^{-\frac{3}{2}} \sum_{i=1}^{d}\left(\frac{\big(h'(y_i) - c_h\big)^2}{\frac{1}{2}h''(y_i)} zz^\top + 4y_i^2 h''(y_i) z_i z_i^\top + 2y_i (h'(y_i) - c_h)\big(zz_i^\top + z_i z^\top\big)\right) \\
&= \frac{\lambda_h}{4}\|x\|_1^{-\frac{3}{2}} J + \frac{1}{4}\|x\|_1^{-\frac{3}{2}} \sum_{i=1}^{d}\left(\frac{h'(y_i) - c_h}{\sqrt{\frac{1}{2}h''(y_i)}} z + 2y_i\sqrt{\tfrac{1}{2}h''(y_i)} z_i\right)\left(\frac{h'(y_i) - c_h}{\sqrt{\frac{1}{2}h''(y_i)}} z + 2y_i\sqrt{\tfrac{1}{2}h''(y_i)} z_i\right)^\top \\
&\quad + \frac{\lambda_h}{4}\|x\|_1^{-\frac{3}{2}} \sum_{i=1}^{d} y_i^2 h''(y_i) z_i z_i^\top,
\end{aligned}$$

and the result follows since each term in the first summation is psd. ∎

B Proof of Main Result

In this section we provide the proof of Theorem 1. In Appendix B.1 we prove useful lemmas which provide us with stability properties of the FTRL iterates. In Appendix B.2 and Appendix B.3 we bound the stability and penalty terms (RHS of Eq. (6) and Eq. (5)) towards proving Theorem 2 in Appendix B.4. We then prove Theorem 1 in Appendix B.5.

B.1 Stability of Iterates

We first establish a technical stability property of the FTRL updates that is crucial for bounding the stability term (Eq. (6)). This property asserts that for every time step t, the clique marginal probabilities induced by p_t are close, up to a constant multiplicative factor, to the clique marginals induced by p_t^+, where $p_t^+ \triangleq \arg\min_{p\in\mathcal{S}_N^\gamma} \{\widehat{L}_t \cdot p + R_t(p)\}$. The proof uses properties of the log-barrier component Φ, and relies on an adaptation of an argument of Jin and Luo [13].

Lemma 7. *For all time steps t and cliques V_k it holds that $p_t^+(V_k) \leq \frac{7}{3}p_t(V_k)$, where $p_t^+ \triangleq \arg\min_{p\in\mathcal{S}_N^\gamma} \big\{\widehat{L}_t \cdot p + R_t(p)\big\}$.*

Proof. We define:

$$\begin{aligned}
F_t(p) &= \widehat{L}_{t-1} \cdot p + R_t(p), \\
F_t^+(p) &= \widehat{L}_t \cdot p + R_t(p),
\end{aligned}$$

so that $p_t = \arg\min_{p\in\mathcal{S}_N^\gamma} \{F_t(p)\}$ and $p_t^+ = \arg\min_{p\in\mathcal{S}_N^\gamma} \big\{F_t^+(p)\big\}$. Note that $\nabla^2\Phi(p)$ is a block diagonal matrix, with the block corresponding to the clique V_k being exactly $\frac{9}{p(V_k)^2} J_{V_k}$ where J_{V_k} is the $|V_k| \times |V_k|$ all-ones matrix. A straightforward calculation then shows that for all $p, p', p'' \in \mathcal{S}_N$ it holds that:

$$\|p' - p''\|^2_{\nabla^2\Phi(p)} = 9\sum_{k=1}^{K} \frac{(p'(V_k) - p''(V_k))^2}{p(V_k)^2}.$$

It suffices to prove that $\|p_t^+ - p_t\|^2_{\nabla^2\Phi(p_t)} \leq 16$. This is because by the calculation we just made, we have $\big(p_t^+(V_k) - p_t(V_k)\big)^2 \leq (\frac{4}{3}p_t(V_k))^2$ which is want we want to prove. It then suffices to show that for any $p' \in \mathcal{S}_N^\gamma$ with $\|p' - p_t\|^2_{\nabla^2\Phi(p_t)} = 16$ we have $F_t^+(p') \geq F_t^+(p_t)$. This is because as an implication of that, p_t^+ which minimizes the convex function F_t^+, must be within the convex set $\{p : \|p - p_t\|^2_{\nabla^2\Phi(p_t)} \le 16\}$. We proceed to lower bound $F_t^+(p')$ as follows:

$$
\begin{aligned}
F_t^+(p') &= F_t^+(p_t) + \nabla F_t^+(p_t)^\mathsf{T}(p' - p_t) + \frac{1}{2}\|p' - p_t\|^2_{\nabla^2 R_t(\xi)} \\
&= F_t^+(p_t) + \nabla F_t(p_t)^\mathsf{T}(p' - p_t) + \widehat{\ell}_t^\mathsf{T}(p' - p_t) + \frac{1}{2}\|p' - p_t\|^2_{\nabla^2 R_t(\xi)} \\
&\ge F_t^+(p_t) + \widehat{\ell}_t^\mathsf{T}(p' - p_t) + \frac{1}{2}\|p' - p_t\|^2_{\nabla^2\Phi(\xi)},
\end{aligned}
$$

where the first equality is a Taylor expansion of F_t^+ around p_t, with ξ being a point between p' and p_t, and the last inequality is due to first-order optimality conditions and the fact that $\nabla^2 R_t(\xi) \ge \nabla^2\Phi(\xi)$ since Ψ is convex. Note that since $\|p' - p_t\|^2_{\nabla^2\Phi(p_t)} = 16$, by the same argument as in the beginning of the proof we conclude that $p'(V_k) \le \frac{7}{3}p_t(V_k)$. Since ξ lies between p_t and p' we conclude the same ratio bound for ξ. We can thus bound the last term as follows:

$$
\begin{aligned}
\frac{1}{2}\|p' - p_t\|^2_{\nabla^2\Phi(\xi)} &= \frac{9}{2}\sum_{k=1}^{K}\frac{(p'(V_k) - p_t(V_k))^2}{(\xi(V_k))^2} \\
&\ge \frac{9}{2\cdot\left(\frac{7}{3}\right)^2}\sum_{k=1}^{K}\frac{(p'(V_k) - p_t(V_k))^2}{p_t(V_k)^2} \\
&= \frac{9}{2\cdot 49}\|p' - p_t\|^2_{\nabla^2\Phi(p_t)} \\
&= \frac{72}{49} \ge 1.
\end{aligned}
$$

It now suffices to show that $\widehat{\ell}_t^\mathsf{T}(p' - p_t) \ge -1$; indeed,

$$
\widehat{\ell}_t^\mathsf{T}(p' - p_t) = \sum_{i\in V(I_t)}\frac{\ell_{t,i}}{p_t(V(I_t))}(p'_{t,i} - p_{t,i}) \ge -\frac{1}{p_t(V(I_t))}\sum_{i\in V(I_t)}\ell_{t,i}p_{t,i} \ge -1,
$$

and the proof is complete. ■

The following lemma showcases another stability property that relates p_t to p_t^+. A corollary of this lemma is that the pseudo-regret of the iterates p_t can only be larger than the pseudo-regret of the iterates p_t^+, and it is used in the proof of Theorem 1 in Appendix B.5.

Lemma 8. *For all time steps t it holds that*

$$
p_t^+ \cdot \widehat{\ell}_t \le p_t \cdot \widehat{\ell}_t,
$$

where $p_t^+ \triangleq \arg\min_{p\in\mathbb{S}_N^\gamma}\{\widehat{L}_t \cdot p + R_t(p)\}$.

Proof. Since p_t^+ is a minimizer of $\widehat{L}_t \cdot p + R_t(p)$ and p_t is a minimizer of $\widehat{L}_{t-1} \cdot p + R_t(p)$, we have:

$$
\begin{aligned}
\widehat{L}_t \cdot p_t^+ + R_t(p_t^+) &\le \widehat{L}_t \cdot p_t + R_t(p_t) \\
&= \widehat{\ell}_t \cdot p_t + \widehat{L}_{t-1} \cdot p_t + R_t(p_t) \\
&\le \widehat{\ell}_t \cdot p_t + \widehat{L}_{t-1} \cdot p_t^+ + R_t(p_t^+),
\end{aligned}
$$

and the claim follows by rearranging terms. ■

B.2 Proof of Lemma 5 (Stability)

We now restate Lemma 5 which bounds the stability term to include extra constants which appear in the bound.

Lemma 5 (restated)**.** *The following holds for all time steps t:*

$$\mathbb{E}\big[(\|\widehat{\ell}_t - \ell_{t,i^\star}\mathbf{1}\|_t^*)^2\big] = 56\sum_{k\neq k^\star}\sqrt{\mathbb{E}\big[p_t(V_k)\big]} + 8\sqrt{\mathbb{E}\big[p_t^+(V_{k^\star}\setminus i^\star)\big]}.$$

Here $\|g\|_t^* = \sqrt{g^\top(\nabla^2\Psi(\tilde{p}_t))^{-1}g}$ is the dual local norm induced by Ψ at $\tilde{p}_t$ for some intermediate point $\tilde{p}_t \in [p_t, p_t^+]$, where $p_t^+ = \arg\min_{p\in\mathcal{S}_N^\gamma}\{\widehat{L}_t\cdot p + R_t(p)\}$.

Proof. By Lemma 2, $\nabla^2\Psi(\tilde{p}_t)$ is lower bounded by a diagonal matrix D_t in which the i'th diagonal entry corresponding to $i\in V_k$ is $\left(2\sqrt{\tilde{p}_t(V_k)}\tilde{p}_{t,i}\right)^{-1}$. Equivalently it holds that $\left(\nabla^2\Psi(\tilde{p}_t)\right)^{-1} \preceq D_t^{-1}$. Using this fact and the fact that $\widehat{\ell}_{t,i} = 0$ for $i\notin V(I_t)$ we have

$$\begin{aligned}
\mathbb{E}\left[(\|\widehat{\ell}_t - \ell_{t,i^\star}\mathbf{1}\|_t^*)^2\right] &= \mathbb{E}\left[\left(\widehat{\ell}_t - \ell_{t,i^\star}\mathbf{1}\right)^\top\left(\nabla^2\Psi(\tilde{p}_t)\right)^{-1}\left(\widehat{\ell}_t - \ell_{t,i^\star}\mathbf{1}\right)\right]\\
&\le 2\mathbb{E}\left[\sum_{k=1}^K\sqrt{\tilde{p}_t(V_k)}\sum_{i\in V_k}\tilde{p}_{t,i}\left(\widehat{\ell}_{t,i}-\ell_{t,i^\star}\right)^2\right]\\
&= 2\mathbb{E}\left[\sqrt{\tilde{p}_t(V(I_t))}\sum_{i\in V(I_t)}\tilde{p}_{t,i}\left(\widehat{\ell}_{t,i}-\ell_{t,i^\star}\right)^2\right] \qquad (7)\\
&\quad + 2\mathbb{E}\left[\sum_{V_k\neq V(I_t)}\sqrt{\tilde{p}_t(V_k)}\sum_{i\in V_k}\tilde{p}_{t,i}(\ell_{t,i^\star})^2\right], \qquad (8)
\end{aligned}$$

where in the final equality we split the sum over cliques into a term for $V(I_t)$ and a sum over the rest of the cliques. We first show that the RHS of Eq. (7) is bounded as follows:

$$\mathbb{E}\left[\sqrt{\tilde{p}_t(V(I_t))}\sum_{i\in V(I_t)}\tilde{p}_{t,i}\left(\widehat{\ell}_{t,i}-\ell_{t,i^\star}\right)^2\right] \le 16\sum_{k\neq k^\star}\sqrt{\mathbb{E}[p(V_k)]} + 4\sqrt{\mathbb{E}\big[p_t^+(V_{k^\star}\setminus i^\star)\big]}.$$

Indeed, due to Lemma 7 and the fact that $\tilde{p}_t$ lies between p_t and p_t^+ it holds that $\tilde{p}_t(V_k)\le 3p_t(V_k)$ for all k. Plugging in the expression for the loss estimator $\widehat{\ell}_t$ we obtain

$$\begin{aligned}
\mathbb{E}\left[\sqrt{\tilde{p}_t(V(I_t))}\sum_{i\in V(I_t)}\tilde{p}_{t,i}\left(\widehat{\ell}_{t,i}-\ell_{t,i^\star}\right)^2\right] &= \mathbb{E}\left[\sqrt{\tilde{p}_t(V(I_t))}\sum_{i\in V(I_t)}\tilde{p}_{t,i}\left(\frac{\ell_{t,i}}{p_t(V(I_t))}-\ell_{t,i^\star}\right)^2\right]\\
&\le 2\mathbb{E}\left[p_t(V(I_t))^{-\frac{3}{2}}\sum_{i\in V(I_t)}\tilde{p}_{t,i}\left(\ell_{t,i}-p_t(V(I_t))\ell_{t,i^\star}\right)^2\right]\\
&= 2\mathbb{E}\left[\sum_{k=1}^K p_t(V_k)^{-\frac{1}{2}}\sum_{i\in V_k}\tilde{p}_{t,i}\left(\ell_{t,i}-p_t(V_k)\ell_{t,i^\star}\right)^2\right],
\end{aligned}$$

where in the last equality we use the law of total expectation and the fact that conditioned on the history up until time step t (including the decision vector p_t), the probability that I_t belongs to the clique V_k is exactly $p_t(V_k)$. In more detail:

$$\begin{aligned}
&\mathbb{E}\left[p_t(V(I_t))^{-\frac{3}{2}}\sum_{i\in V(I_t)}\tilde{p}_{t,i}\left(\ell_{t,i}-p_t(V(I_t))\ell_{t,i^\star}\right)^2\right]\\
&= \mathbb{E}\left[\mathbb{E}_t\left[p_t(V(I_t))^{-\frac{3}{2}}\sum_{i\in V(I_t)}\tilde{p}_{t,i}\left(\ell_{t,i}-p_t(V(I_t))\ell_{t,i^\star}\right)^2\right]\right]\\
&= \mathbb{E}\left[\sum_{k=1}^K\Pr[I_t\in V_k\mid h_t]\cdot\mathbb{E}_t\left[p_t(V_k)^{-\frac{3}{2}}\sum_{i\in V_k}\tilde{p}_{t,i}\left(\ell_{t,i}-p_t(V_k)\ell_{t,i^\star}\right)^2\right]\right]
\end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}\left[\sum_{k=1}^{K} p_t(V_k)\cdot \mathbb{E}_t\left[p(V_k)^{-\frac{3}{2}} \sum_{i\in V_k} \tilde{p}_{t,i}\big(\ell_{t,i} - p_t(V_k)\ell_{t,i^\star}\big)^2\right]\right] \\
&= \mathbb{E}\left[\mathbb{E}_t\left[\sum_{k=1}^{K} p_t(V_k)^{-\frac{1}{2}} \sum_{i\in V_k} \tilde{p}_{t,i}\big(\ell_{t,i} - p_t(V_k)\ell_{t,i^\star}\big)^2\right]\right] \\
&= \mathbb{E}\left[\sum_{k=1}^{K} p_t(V_k)^{-\frac{1}{2}} \sum_{i\in V_k} \tilde{p}_{t,i}\big(\ell_{t,i} - p_t(V_k)\ell_{t,i^\star}\big)^2\right],
\end{aligned}
$$

where h_t denotes the history up to and including the choice of p_t at time step t (not including the choice of I_t), and in the fourth equality we use linearity of expectation and the fact that $p_t(V_k)$ is constant when conditioned on h_t. We proceed to bound the above term, while splitting the sum over cliques into a term for $V_{k^\star}$ and a sum for all of the other cliques:

$$
\begin{aligned}
&\mathbb{E}\left[\sum_{k=1}^{K} p_t(V_k)^{-\frac{1}{2}} \sum_{i\in V_k} \tilde{p}_{t,i}\big(\ell_{t,i} - p_t(V_k)\ell_{t,i^\star}\big)^2\right] \\
&\le \mathbb{E}\left[\sum_{k\neq k^\star} p_t(V_k)^{-\frac{1}{2}}\tilde{p}_t(V_k)\right] + \mathbb{E}\left[p_t(V_{k^\star})^{-\frac{1}{2}}\Bigg(\sum_{i\in V_{k^\star}, i\neq i^\star} \tilde{p}_{t,i} + \tilde{p}_{t,i^\star}(1-p_t(V_{k^\star}))^2\Bigg)\right] \\
&\le 3\mathbb{E}\left[\sum_{k\neq k^\star}\sqrt{p_t(V_k)}\right] + 2\mathbb{E}\left[\tilde{p}_t(V_{k^\star})^{-\frac{1}{2}}\tilde{p}_t\big(V_{k^\star}\setminus i^\star\big)\right] + 3\mathbb{E}\left[(1-p_t(V_{k^\star}))^2\right] \\
&\le 6\mathbb{E}\left[\sum_{k\neq k^\star}\sqrt{p_t(V_k)}\right] + 2\mathbb{E}\left[\sqrt{\tilde{p}_t(V_{k^\star}\setminus i^\star)}\right] \\
&\le 8\mathbb{E}\left[\sum_{k\neq k^\star}\sqrt{p_t(V_k)}\right] + 2\mathbb{E}\left[\sqrt{p_t^+(V_{k^\star}\setminus i^\star)}\right] \\
&\le 8\sum_{k\neq k^\star}\sqrt{\mathbb{E}[p_t(V_k)]} + 2\sqrt{\mathbb{E}\left[p_t^+(V_{k^\star}\setminus i^\star)\right]},
\end{aligned}
$$

where in the last inequality we used Jensen's inequality. We now proceed to bound the RHS of Eq. (8):

$$
\begin{aligned}
\mathbb{E}\left[\sum_{V_k\neq V(I_t)}\sqrt{\tilde{p}_t(V_k)}\sum_{i\in V_k}\tilde{p}_{t,i}(\ell_{t,i^\star})^2\right] &\le \mathbb{E}\left[\sum_{V_k\neq V(I_t)}\tilde{p}_t(V_k)^{\frac{3}{2}}\right] \\
&\le 6\mathbb{E}\left[\sum_{V_k\neq V(I_t)} p_t(V_k)^{\frac{3}{2}}\right] \qquad (9) \\
&= 6\mathbb{E}\left[\sum_{k=1}^{K}(1-p_t(V_k))p_t(V_k)^{\frac{3}{2}}\right] \qquad (10) \\
&\le 12\mathbb{E}\left[\sum_{k\neq k^\star}\sqrt{p_t(V_k)}\right] \\
&\le 12\sum_{k\neq k^\star}\sqrt{\mathbb{E}[p_t(V_k)]},
\end{aligned}
$$

where in Eq. (9) we use Lemma 7 and the fact that $\tilde{p}_t$ lies between p_t and p_t^+, in Eq. (10) we use the fact that the probability of the clique V_k not to be chosen at time step t is $1 - p_t(V_k)$ and the last line uses Jensen's inequality. Combining the two bounds, we conclude the proof. ■

B.3 Proof of Lemma 4 (Penalty)

In this section we restate Lemma 4 which bounds the penalty term to include the extra constants and poly-log factors.

Lemma 4 (restated)**.** *The penalty term described in the RHS of Eq.* (5) *is bounded by*

$$9K\log\frac{1}{\gamma}+5\log^2\frac{1}{\gamma}\sum_{t=1}^{T}\sum_{k\neq k^\star}\sqrt{\frac{p_t(V_k)}{t}}+2\log\frac{1}{\gamma}\sum_{t=1}^{T}\sqrt{\frac{p_t(V_{k^\star}\setminus i^\star)}{t}}, \tag{11}$$

where $p_i^\gamma=\begin{cases}\gamma & i\neq i^\star\\ 1-(N-1)\gamma & i=i^\star\end{cases}$ *for all* $i\in[N]$ *and* $\frac{1}{\eta_0}\triangleq 0$.

Proof. Noting that $\Phi(\cdot)\geq 0$ we can bound the first term as follows:

$$\Phi(p^\gamma)-\Phi(p_1)\leq\Phi(p^\gamma)\leq 9K\log\frac{1}{\gamma}.$$

Continuing with the second part of the penalty term, note that by definition of p^γ we have $p^\gamma(V_{k^\star})\geq p_t(V_{k^\star})$ for all t. Also note that $\Psi(p^\gamma)\leq -2\left(\log^2\frac{1}{\gamma}+1\right)\sqrt{p^\gamma(V_{k^\star})}$. We then have

$$\begin{aligned}\sum_{t=1}^{T}\left(\frac{1}{\eta_t}-\frac{1}{\eta_{t-1}}\right)(\Psi(p^\gamma)-\Psi(p_t)) &\leq \sum_{t=1}^{T}\left(\sqrt{t}-\sqrt{t-1}\right)\Bigg(2\left(\log^2\frac{1}{\gamma}+1\right)\sum_{k=1}^{K}\sqrt{p_t(V_k)}\\ &\quad+\sum_{k=1}^{K}\frac{1}{\sqrt{p_t(V_k)}}\sum_{i\in V_k}p_{t,i}\log\frac{p_t(V_k)}{p_{t,i}}-2\left(\log^2\frac{1}{\gamma}+1\right)\sqrt{p^\gamma(V_{k^\star})}\Bigg)\\ &\leq 2\left(\log^2\frac{1}{\gamma}+1\right)\sum_{t=1}^{T}\frac{1}{\sqrt{t}}\sum_{k\neq k^\star}\sqrt{p_t(V_k)}\\ &\quad+\log\frac{1}{\gamma}\sum_{t=1}^{T}\frac{1}{\sqrt{t}}\sum_{k\neq k^\star}\sqrt{p_t(V_k)}\\ &\quad+\sum_{t=1}^{T}\frac{1}{\sqrt{t\cdot p_t(V_{k^\star})}}\sum_{i\in V_{k^\star}}p_{t,i}\log\frac{p_t(V_{k^\star})}{p_{t,i}}\\ &\leq 5\log^2\frac{1}{\gamma}\sum_{t=1}^{T}\frac{1}{\sqrt{t}}\sum_{k\neq k^\star}\sqrt{p_t(V_k)}\\ &\quad+\sum_{t=1}^{T}\frac{1}{\sqrt{t\cdot p_t(V_{k^\star})}}\sum_{i\in V_{k^\star}}p_{t,i}\log\frac{p_t(V_{k^\star})}{p_{t,i}},\end{aligned} \tag{12}$$

where the second inequality follows from the fact that $\frac{1}{\sqrt{t}}-\frac{1}{\sqrt{t-1}}\leq\frac{1}{\sqrt{t}}$ and that $p_{t,i}\geq\gamma$ for all t and i. It is left to bound the final term. Using the inequality $\log x\leq x-1$ for all $x>0$ we have

$$\begin{aligned}\sum_{i\in V_{k^\star}}p_{t,i}\log\frac{p_t(V_{k^\star})}{p_{t,i}} &= \sum_{i\in V_{k^\star}\setminus i^\star}p_{t,i}\log\frac{p_t(V_{k^\star})}{p_{t,i}}+p_{t,i^\star}\log\frac{p_t(V_{k^\star})}{p_{t,i^\star}}\\ &\leq \log\frac{1}{\gamma}\sum_{i\in V_{k^\star}\setminus i^\star}p_{t,i}+p_{t,i^\star}\left(\frac{p_t(V_{k^\star})}{p_{t,i^\star}}-1\right)\\ &= \left(\log\frac{1}{\gamma}+1\right)p_t\left(V_{k^\star}\setminus i^\star\right)\\ &\leq 2\log\frac{1}{\gamma}p_t\left(V_{k^\star}\setminus i^\star\right).\end{aligned}$$

Plugging this bound into Eq. (12) while using the fact that $\frac{p_t(V_{k^\star}\setminus i^\star)}{\sqrt{p_t(V_{k^\star})}}\leq\sqrt{p_t(V_{k^\star}\setminus i^\star)}$ completes the proof. ■

B.4 Proof of Theorem 2

In order to prove Theorem 2 we make use of the following simple claim which asserts that the pseudo-regret is bounded up to an additive constant factor by the regret with respect to some probability vector in $\mathcal{S}_N^\gamma$.

Lemma 9. *For all $\gamma \in [0, \frac{1}{N}]$ and $i^\star \in [N]$ the following holds:*

$$\mathbb{E}\left[\sum_{t=1}^T p_t \cdot \widehat{\ell}_t - \mathbf{e}_{i^\star} \cdot \sum_{t=1}^T \widehat{\ell}_t\right] \le \mathbb{E}\left[\sum_{t=1}^T p_t \cdot \widehat{\ell}_t - p^\gamma \cdot \sum_{t=1}^T \widehat{\ell}_t\right] + \gamma T N,$$

where $p_i^\gamma = \begin{cases}\gamma & i \neq i^\star \\ 1-(N-1)\gamma & i = i^\star\end{cases} \quad \forall i \in [N].$

Proof. Fix $\gamma \in [0, \frac{1}{N}]$ and $i^\star \in [N]$. Note that $\mathbf{e}_{i^\star} = p^\gamma - v$ where v is defined as follows:

$$v_i = \begin{cases}\gamma & i \neq i^\star \\ -(N-1)\gamma & i = i^\star\end{cases} \quad \forall i \in [N].$$

This observation gives us the following:

$$\begin{aligned}\mathbb{E}\left[\sum_{t=1}^T p_t \cdot \widehat{\ell}_t - \mathbf{e}_{i^\star} \cdot \sum_{t=1}^T \widehat{\ell}_t\right] &= \mathbb{E}\left[\sum_{t=1}^T p_t \cdot \ell_t - \mathbf{e}_{i^\star} \cdot \sum_{t=1}^T \ell_t\right] \\ &= \mathbb{E}\left[\sum_{t=1}^T p_t \cdot \ell_t - p^\gamma \cdot \sum_{t=1}^T \ell_t\right] + v \cdot \mathbb{E}\left[\sum_{t=1}^T \ell_t\right] \\ &= \mathbb{E}\left[\sum_{t=1}^T p_t \cdot \widehat{\ell}_t - p^\gamma \cdot \sum_{t=1}^T \widehat{\ell}_t\right] + v \cdot \mathbb{E}\left[\sum_{t=1}^T \ell_t\right],\end{aligned}$$

where the first equality is due to the fact that $\widehat{\ell}_t$ is an unbiased estimator of ℓ_t. We bound the last term using the expression for v:

$$v \cdot \sum_{t=1}^T \ell_t = \sum_{t=1}^T \left[\sum_{i \neq i^\star} \gamma \ell_{t,i} - (N-1)\gamma \ell_{t,i^\star}\right] \le \gamma T N,$$

where in the last inequality we use the fact that the losses are bounded in $[0, 1]$. ■

We will also make use of general FTRL regret bound given by Theorem 3 (which we prove in Appendix C) together with the stability and penalty bounds shown in the previous sections. Theorem 2 is restated here in the precise form proved below.

Theorem 2 (restated)**.** *Algorithm 1 attains the following regret bound, regardless of the corruption level, for $NT \ge 3^{11}$:*

$$\begin{aligned}\mathcal{R}_T \le\; & 9K\log(NT) + 6\log^2(NT) \sum_{t=1}^T \sum_{k \neq k^\star} \sqrt{\frac{\mathbb{E}[p_t(V_k)]}{t}} \\ & + 2\log(NT) \sum_{t=1}^T \sqrt{\frac{\mathbb{E}[p_t(V_{k^\star} \setminus i^\star)]}{t}} + 16 \sum_{t=1}^T \sqrt{\frac{\mathbb{E}[p_t^+(V_{k^\star} \setminus i^\star)]}{t}}. \end{aligned} \tag{13}$$

Proof. Note that due to Lemma 9 it suffices to bound $\mathbb{E}\big[\sum_{t=1}^T (p_t - p^\gamma) \cdot \widehat{\ell}_t\big]$ where p^γ is defined by

$$p_i^\gamma = \begin{cases}\gamma & i \neq i^\star \\ 1-(N-1)\gamma & i = i^\star\end{cases} \quad \forall i \in [N],$$

since it can only be larger than the pseudo-regret by an additive constant. Using Theorem 3 and then bounding the penalty and stability terms using Lemma 4 and Lemma 5 we obtain

$$\mathcal{R}_T \le 9K\log(NT) + \left(5\log^2(NT) + 112\right) \sum_{t=1}^T \sum_{k \neq k^\star} \sqrt{\frac{\mathbb{E}[p_t(V_k)]}{t}}$$

$$+ 2\log(NT)\sum_{t=1}^{T}\sqrt{\frac{\mathbb{E}[p_t(V_{k^\star}\setminus i^\star)]}{t}} + 16\sum_{t=1}^{T}\sqrt{\frac{\mathbb{E}[p_t^+(V_{k^\star}\setminus i^\star)]}{t}}$$

$$\le 9K\log(NT) + 6\log^2(NT)\sum_{t=1}^{T}\sum_{k\neq k^\star}\sqrt{\frac{\mathbb{E}[p_t(V_k)]}{t}}$$

$$+ 2\log(NT)\sum_{t=1}^{T}\sqrt{\frac{\mathbb{E}[p_t(V_{k^\star}\setminus i^\star)]}{t}} + 16\sum_{t=1}^{T}\sqrt{\frac{\mathbb{E}[p_t^+(V_{k^\star}\setminus i^\star)]}{t}},$$

where the last inequality holds for $NT \ge 3^{11}$. ■

B.5 Proof of Theorem 1 (Main)

We can now provide a proof of our main result given in Theorem 1, restated here more precisely.

Theorem 1 (restated)**.** *Algorithm 1 attains the following expected pseudo-regret bound in the C-corrupted stochastic setting, for $NT \ge 3^{11}$:*

$$\mathcal{R}_T \le 184\log^2(NT)\cdot\min\left\{\sqrt{KT}, \log^2(NT)\sum_{k:\Delta_k>0}\frac{\log T}{\Delta_k} + \sqrt{C\sum_{k:\Delta_k>0}\frac{\log T}{\Delta_k}}\right\}.$$

Proof. We first prove the following:

$$\mathcal{R}_T \le 184\log^4(NT)\sum_{k:\Delta_k>0}\frac{\log T}{\Delta_k} + 28\log^2(NT)\sqrt{C\sum_{k:\Delta_k>0}\frac{\log T}{\Delta_k}}.$$

We proceed bounding the RHS of Eq. (13). For all $B, z > 0$ we have

$$\begin{aligned}
B\sum_{t=1}^{T}\left(\sum_{k\neq k^\star}\sqrt{\frac{\mathbb{E}[p_t(V_k)]}{t}} + \sqrt{\frac{\mathbb{E}[p_t(V_{k^\star}\setminus i^\star)]}{t}}\right) &\le B^2\cdot z\sum_{t=1}^{T}\sum_{k:\Delta_k>0}\frac{1}{2t\Delta_k} + \frac{1}{2z}\sum_{t=1}^{T}\sum_{i=1}^{N}\mathbb{E}[p_{t,i}]\delta_i \\
&\le B^2\cdot z\sum_{k:\Delta_k>0}\frac{\log T}{\Delta_k} + \frac{1}{2z}\sum_{t=1}^{T}\sum_{i=1}^{N}\mathbb{E}[p_{t,i}]\delta_i \\
&\le B^2\cdot z\sum_{k:\Delta_k>0}\frac{\log T}{\Delta_k} + \frac{1}{2z}(\mathcal{R}_T + 2C), \qquad (14)
\end{aligned}$$

where the first inequality is due to Young's inequality and the fact that $\Delta_k \le \delta_i$ for all $i \in V_k$, the second inequality is since $\sum_{t=1}^{T}(1/t) \le 2\log T$ and the last inequality is due to the following simple observation which follows from the definition of corruption:

$$\begin{aligned}
\mathbb{E}\left[\sum_{t=1}^{T}\sum_{i=1}^{N}p_{t,i}(\tilde{\ell}_{t,i} - \tilde{\ell}_{t,i^\star})\right] &\le \mathbb{E}\left[\sum_{t=1}^{T}\sum_{i=1}^{N}p_{t,i}(\ell_{t,i} - \ell_{t,i^\star})\right] + 2\mathbb{E}\left[\sum_{t=1}^{T}\|\ell_t - \tilde{\ell}_t\|_\infty\right] \\
&= \mathbb{E}\left[\sum_{t=1}^{T}\sum_{i=1}^{N}p_{t,i}(\ell_{t,i} - \ell_{t,i^\star})\right] + 2C.
\end{aligned}$$

Setting $B = 6\log^2(NT)$ gives a bound on the second term in the RHS of Eq. (13). Similarly, we have

$$\begin{aligned}
16\sum_{t=1}^{T}\sqrt{\frac{1}{t}\mathbb{E}[p_t^+(V_{k^\star}\setminus i^\star)]} &\le 256z\sum_{k:\Delta_k>0}\frac{\log T}{\Delta_k} + \frac{1}{2z}\sum_{t=1}^{T}\sum_{i=1}^{N}\mathbb{E}[p_{t,i}^+]\delta_i \\
&\le 256z\sum_{k:\Delta_k>0}\frac{\log T}{\Delta_k} + \frac{C}{z} + \frac{1}{2z}\mathbb{E}\left[\sum_{t=1}^{T}\sum_{i=1}^{N}p_{t,i}^+\cdot(\ell_{t,i} - \ell_{t,i^\star})\right]. \qquad (15)
\end{aligned}$$

We now use Lemma 8 to bound the rightmost term of Eq. (15) as follows:

$$\begin{aligned}\mathbb{E}\left[\sum_{t=1}^{T}\sum_{i=1}^{N} p_{t,i}^{+}\cdot\left(\ell_{t,i}-\ell_{t,i^\star}\right)\right] &= \mathbb{E}\left[\sum_{t=1}^{T} p_t^{+}\cdot\left(\mathbb{E}_t[\widehat{\ell}_t]-\ell_{t,i^\star}\mathbf{1}\right)\right]\\ &\le \mathbb{E}\left[\sum_{t=1}^{T} p_t\cdot\left(\mathbb{E}_t[\widehat{\ell}_t]-\ell_{t,i^\star}\mathbf{1}\right)\right]\\ &= \mathbb{E}\left[\sum_{t=1}^{T}\sum_{i=1}^{N} p_{t,i}\cdot\left(\ell_{t,i}-\ell_{t,i^\star}\right)\right]\\ &= \mathcal{R}_T,\end{aligned}$$

where we used the fact that $\widehat{\ell}_t$ is an unbiased estimator for ℓ_t. We can conclude that

$$16\sum_{t=1}^{T}\sum_{k:\Delta_k>0}\sqrt{\frac{1}{t}\mathbb{E}[p_t^{+}(V_k)]} \le 256\sum_{k:\Delta_k>0}\frac{\log T}{\Delta_k}+\frac{1}{2z}(\mathcal{R}_T+2C). \tag{16}$$

Using Theorem 2 and combining the bounds from Eq. (14) and Eq. (16) we obtain

$$\begin{aligned}\mathcal{R}_T &\le 9K\log(NT)+\left(36\log^4(NT)+256\right)z\sum_{k:\Delta_k>0}\frac{\log T}{\Delta_k}+\frac{1}{z}\mathcal{R}_T+\frac{2C}{z}\\ &\le \left(45\log^4(NT)+256\right)z\sum_{k:\Delta_k>0}\frac{\log T}{\Delta_k}+\frac{1}{z}\mathcal{R}_T+\frac{2C}{z}\\ &\le 46\log^4(NT)z\sum_{k:\Delta_k>0}\frac{\log T}{\Delta_k}+\frac{1}{z}\mathcal{R}_T+\frac{2C}{z},\end{aligned}$$

where the second inequality is since $K \le 1+\sum_{k:\Delta_k>0} 1/\Delta_k$ and the last inequality holds since $NT \ge 3^4$. Rearranging and simplifying we obtain

$$\mathcal{R}_T \le 2U+(z-1)U+\frac{2C+U}{z-1},$$

where we denote $U = 46\log^4(NT)\sum_{k:\Delta_k>0}\log T/\Delta_k$ for simplicity. We now choose z which minimizes the bound, by setting $z = 1+\sqrt{\frac{U+2C}{U}}$. This gives us

$$\begin{aligned}\mathcal{R}_T &\le 2U+2\sqrt{U(U+2C)}\\ &\le 4U+4\sqrt{UC}\\ &\le 184\log^2(NT)\sum_{k:\Delta_k>0}\frac{\log T}{\Delta_k}+28\log^2(NT)\sqrt{C\sum_{k:\Delta_k>0}\frac{\log T}{\Delta_k}},\end{aligned}$$

which concludes the first part of the proof. We now show that

$$\mathcal{R}_T \le 28\log^2(NT)\sqrt{KT}.$$

We again use Theorem 2 and also the fact that $p_t^{+}(V_k) \le \frac{7}{3}p_t(V_k)$ by Lemma 7, to obtain

$$\begin{aligned}\mathcal{R}_T &\le 9K\log(NT)+\left(6\log^2(NT)+32\right)\sum_{t=1}^{T}\frac{1}{\sqrt{t}}\sum_{k=1}^{K}\sqrt{p_t(V_k)}\\ &\le 9K\log(NT)+7\log^2(NT)\sum_{t=1}^{T}\frac{1}{\sqrt{t}}\sum_{k=1}^{K}\sqrt{p_t(V_k)},\end{aligned}$$

where the inequality holds since $NT \ge 3^6$. We conclude the proof via the following straightforward calculation:

$$\sum_{t=1}^{T}\frac{1}{\sqrt{t}}\sum_{k=1}^{K}\sqrt{p_t(V_k)} \le \sqrt{K}\sum_{t=1}^{T}\frac{1}{\sqrt{t}} \le 2\sqrt{KT},$$

where we used Jensen's inequality and the fact that $\sum_{t=1}^{T}(1/\sqrt{t}) \le 2\sqrt{T}$. We obtained two regret bounds and thus the minimum of the two holds, which concludes the proof. ■

C Refined Regret Bound for FTRL

Consider the FTRL framework which generates predictions $w_1, w_2, \ldots, w_T \in \mathcal{W}$ given a sequence of arbitrary loss vectors $g_1, g_2, \ldots, g_T$ and a sequence of regularization functions $H_1, H_2, \ldots, H_T$. The following gives a general regret bound which we use in order to prove Theorem 2.

Theorem 3. *Suppose $H_t = \eta_t^{-1}\psi + \phi$ for twice-differentiable and convex functions ψ and ϕ, ψ being strictly convex. Let $w_t^+ = \arg\min_{w\in\mathcal{W}}\{w \cdot \sum_{s=1}^t g_s + H_t(w)\}$. Then there exists a sequence of points $\tilde{w}_t \in [w_t, w_t^+]$ such that, for all $w^* \in \mathcal{W}$:*

$$\sum_{t=1}^T g_t \cdot (w_t - w^\star) \leq \phi(w^\star) - \phi(w_1) + \sum_{t=1}^T \left(\frac{1}{\eta_t} - \frac{1}{\eta_{t-1}}\right)(\psi(w^\star) - \psi(w_t)) + 2\sum_{t=1}^T \eta_t \left(\|g_t\|_t^*\right)^2.$$

Here $\|g\|_t = \sqrt{g^\top \nabla^2 \psi(\tilde{w}_t) g}$ is the local norm induced by ψ at $\tilde{w}_t$, and $\|\cdot\|_t^*$ is its dual. Here we also define $1/\eta_0 \triangleq 0$.

Proof. We directly follow an analysis by Jin and Luo [13], and include the details for completeness. For simplicity we denote $G_t = \sum_{s=1}^t g_s$. We make the following definitions:

$$F_t(w) = w \cdot G_{t-1} + H_t(w),$$
$$F_t^+(w) = w \cdot G_t + H_t(w),$$

such that $w_t = \arg\min_{w\in\mathcal{W}}\{F_t(w)\}$ and $w_t^+ = \arg\min_{w\in\mathcal{W}}\{F_t^+(w)\}$. Fix $w^\star \in \mathcal{W}$. We note that the regret of FTRL with respect to $w^\star$ has the following decomposition:

$$\sum_{t=1}^T g_t \cdot \left(w_t - w^\star\right) = \sum_{t=1}^T \left(w_t \cdot g_t + F_t(w_t) - F_t^+(w_t^+)\right) + \sum_{t=1}^T \left(F_t^+(w_t^+) - F_t(w_t) - w^\star \cdot g_t\right).$$

We first show that for all time steps t it holds that

$$w_t \cdot g_t + F_t(w_t) - F_t^+(w_t^+) \leq 2\eta_t \left(\|g_t\|_t^*\right)^2. \tag{17}$$

We lower bound $w_t \cdot g_t + F_t(w_t) - F_t^+(w_t^+)$ as follows:

$$\begin{aligned} w_t \cdot g_t + F_t(w_t) - F_t^+(w_t^+) &= w_t \cdot G_t + H_t(w_t) - F_t^+(w_t^+) \\ &= F_t^+(w_t) - F_t^+(w_t^+) \\ &= \nabla F_t^+(w_t^+) \cdot \left(w_t - w_t^+\right) + \frac{1}{2}\|w_t - w_t^+\|^2_{\nabla^2 H_t(\tilde{w}_t)} \\ &\geq \frac{1}{2}\|w_t - w_t^+\|^2_{\nabla^2 H_t(\tilde{w}_t)} \\ &\geq \frac{1}{2}\eta_t^{-1}\|w_t - w_t^+\|_t^2, \end{aligned}$$

where the third line is a Taylor expansion of F_t^+ around w_t^+, with $\tilde{w}_t$ being a point between w_t and w_t^+, in the second to last line we use a first-order optimality condition of w_t^+, and in the last line we use the fact that $\nabla^2 H_t \succeq \eta_t^{-1}\nabla^2\psi$. We now upper bound $w_t \cdot g_t + F_t(w_t) - F_t^+(w_t^+)$ as follows:

$$\begin{aligned} w_t \cdot g_t + F_t(w_t) - F_t^+(w_t^+) &= \left(w_t - w_t^+\right) \cdot g_t + F_t(w_t) - F_t(w_t^+) \\ &\leq \left(w_t - w_t^+\right) \cdot g_t \\ &\leq \left(\sqrt{\eta_t^{-1}}\|w_t - w_t^+\|_t\right)\left(\sqrt{\eta_t}\|g_t\|_t^*\right) \\ &= \|w_t - w_t^+\|_t \cdot \|g_t\|_t^*, \end{aligned}$$

where in the first inequality we use the fact that w_t is the minimizer of F_t and the second inequality is an application of Hölder's inequality. Combining the lower and upper bounds gives us Eq. (17). Next we show that

$$\sum_{t=1}^T \left(F_t^+(w_t^+) - F_t(w_t) - w^\star \cdot g_t\right) \leq \phi(w^\star) - \phi(w_1) + \sum_{t=1}^T \left(\frac{1}{\eta_t} - \frac{1}{\eta_{t-1}}\right)(\psi(w^\star) - \psi(w_t)). \tag{18}$$

We bound the LHS of Eq. (18) as follows:

$$\begin{aligned}
&\sum_{t=1}^{T}\big(F_t^+(w_t^+) - F_t(w_t) - w^\star \cdot g_t\big) \\
&\le -F_1(w_1) + \sum_{t=2}^{T}\big(F_{t-1}^+(w_t) - F_t(w_t)\big) + F_T^+(w_T^+) - w^\star \cdot G_T \\
&\le -F_1(w_1) + \sum_{t=2}^{T}\big(F_{t-1}^+(w_t) - F_t(w_t)\big) + F_T^+(w^\star) - w^\star \cdot G_T \\
&= -H_1(w_1) - \sum_{t=2}^{T}\left(\frac{1}{\eta_t} - \frac{1}{\eta_{t-1}}\right)\psi(w_t) + H_T(w^\star) \\
&= -\eta_1^{-1}\psi(w_1) - \phi(w_1) - \sum_{t=2}^{T}\left(\frac{1}{\eta_t} - \frac{1}{\eta_{t-1}}\right)\psi(w_t) + \eta_T^{-1}\psi(w^\star) + \phi(w^\star) \\
&= \phi(w^\star) - \phi(w_1) + \sum_{t=1}^{T}\left(\frac{1}{\eta_t} - \frac{1}{\eta_{t-1}}\right)\big(\psi(w^\star) - \psi(w_t)\big),
\end{aligned}$$

where in the first and second inequalities we use the optimality of w_t^+. Combining Eq. (17) and Eq. (18) we conclude the proof. ■

Proof of Lemma 3. Fix any $p^\gamma \in \mathcal{S}_N^\gamma$. The lemma follows immediately by applying Theorem 3 to Algorithm 1 with the regularizations $R_1, R_2, \dots, R_T$ and the shifted loss estimators $\widehat{\ell_t} - \ell_{t,i^\star}\mathbf{1}$, while noting that constant shifts in the loss estimators do not change the algorithm whatsoever. ■