A The principle of symmetry breaking

A.1 $S_k \times S_d$-invariance properties of (2)

We show that $\mathcal{L}$ is $S_k \times S_d$-invariant in its first parameter. The proof is a straightforward adaption of [3, Section 4.1]. First, we make the dependence of $\mathcal{L}$ on the target weight $d \times d$-matrix V explicit:

$$\overline{\mathcal{L}}(W, \boldsymbol{\alpha}; V, \boldsymbol{\beta}) \coloneqq \frac{1}{2}\mathbb{E}_{\boldsymbol{x}\sim\mathcal{N}(\mathbf{0},I_d)}\left[\left(\sum_{i=1}^{k}\alpha_i\varphi(\langle \boldsymbol{w}_i, \boldsymbol{x}\rangle) - \sum_{i=1}^{d}\beta_i\varphi(\langle \boldsymbol{v}_i, \boldsymbol{x}\rangle)\right)^2\right], \tag{11}$$

Next, we observe that for any $\pi \in S_k, \rho \in S_d$ and $U \in \mathrm{O}(d)$, the group of all $d \times d$-orthogonal matrices, we have

$$\overline{\mathcal{L}}(W, \boldsymbol{\alpha}; V, \boldsymbol{\beta}) = \overline{\mathcal{L}}(P_\pi W, \boldsymbol{\alpha}; V, \boldsymbol{\beta}) = \overline{\mathcal{L}}(W, \boldsymbol{\alpha}; P_\rho V, \boldsymbol{\beta}), \tag{12}$$

$$\overline{\mathcal{L}}(W, \boldsymbol{\alpha}; V, \boldsymbol{\beta}) = \overline{\mathcal{L}}(WU, \boldsymbol{\alpha}; VU, \boldsymbol{\beta}), \tag{13}$$

where the last equality follows by the $\mathrm{O}(d)$-invariance of the standard multivariate Gaussian distribution. Therefore, for any $\rho \in S_d$ and $U \in \mathrm{O}(d)$ such that $V = P_\rho V U^\top$ and any $\pi \in S_k$, we have

$$\begin{aligned}\overline{\mathcal{L}}(W, \boldsymbol{\alpha}, V, \boldsymbol{\beta}) &= \overline{\mathcal{L}}(W, \boldsymbol{\alpha}, P_\rho V U^\top, \boldsymbol{\beta}) \overset{(12)}{=} \overline{\mathcal{L}}(W, \boldsymbol{\alpha}, V U^\top, \boldsymbol{\beta}) \overset{(13)}{=} \overline{\mathcal{L}}(WU, \boldsymbol{\alpha}, V U^\top U, \boldsymbol{\beta}) \\ &= \overline{\mathcal{L}}(WU, \boldsymbol{\alpha}, V, \boldsymbol{\beta}) \overset{(12)}{=} \overline{\mathcal{L}}(P_\pi WU, \boldsymbol{\alpha}, V, \boldsymbol{\beta}).\end{aligned}$$

In particular, for $V = I_d$, we have $V = P_\pi V P_\pi^\top$ for any $\pi \in S_d$, thus $\mathcal{L}(W, \boldsymbol{\alpha}) = \overline{\mathcal{L}}(W, \boldsymbol{\alpha}, I_d, \boldsymbol{\beta})$ is $S_k \times S_d$-invariant w.r.t. W. Note that here we do not exploit the rotational invariance of the standard Gaussian distribution, but rather its invariance to permutations. Hence, the same $S_k \times S_d$-invariance holds for any product distribution if $V = I_d$. Indeed, critical points admit maximal isotropy types also when the input distribution is $\mathcal{D} = \mathcal{U}([-1, 1]^d)$ (but not when $\mathcal{D} = \mathcal{U}([0, 2]^d)$).

A.2 Examples of minima for Problem (2)

We display several examples for optimization problem (2) with $k = d = 10$ obtained by running SGD until the gradient norm is driven below $1\mathrm{e}{-8}$.

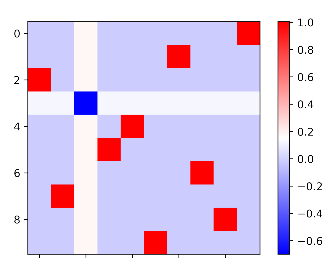

Figure 3: A spurious minimum of isotropy (conjugated to) $\Delta(S_{d-1} \times S_1)$ of (2) with $k = d = 10$. The objective value is ≈ 0.018.

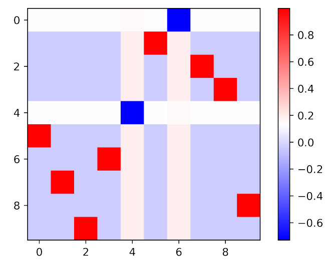

Figure 4: A spurious minimum of isotropy (conjugated to) $\Delta(S_{d-2} \times S_2)$ of (2) with $k = d = 10$. The objective value is ≈ 0.035.

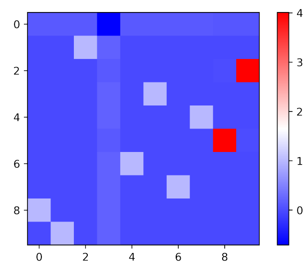

Figure 5: A spurious minimum of (2) with $k = d = 10$, where $V = \text{Diag}(1, \ldots, 1, 2, 2)$ and $\beta = (1, \ldots, 1, 2, 2)$. The symmetry of the minimum adapt to that of the global minimizer V.

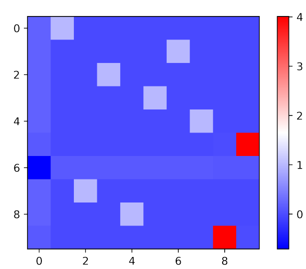

Figure 6: A spurious minimum of (2) with $k = d = 10$, where the $V = \text{Diag}(1, \ldots, 1, 2, 2)$ and $\beta = (1, \ldots, 1, 2, 2)$. The symmetry of the minimum adapt to that of the global minimizer V.

B Counting multiplicity of families of minima

The computation of the multiplicity of minima is based on the orbit-stabilizer theorem. Instantiating this theorem to the natural action of $S_k \times S_d$ (i.e., row- and column- permutation. See Section D.1 below for a formal introduction of group action) yields

$$\text{Multiplicity}(W) = \frac{|S_k \times S_d|}{|\text{Iso}(W)|}.$$

Observing that $|S_k \times S_d| = d!\,k!\,,\ \ |\Delta S_d| = d!\,,\ \ |\Delta(S_{d-1} \times S_1)| = (d-1)!\,,\ \ |\Delta(S_{d-2} \times S_2)| = (d-2)!\,2!$ and $|\Delta(S_{d-3} \times S_3)| = (d-3)!\,3!$ gives the multiplicities stated in Theorem 1.

C Gradient expressions

In the sequel, we provide explicit expressions for $\nabla\mathcal{L}$ (defined in (2)) restricted the fixed point spaces $\mathcal{W}_{d-1}, \mathcal{W}_{d-2}, \mathcal{W}_{d-3}$, which naturally extend Definition 10 as follows

$$\mathcal{W}_p \coloneqq \{W \in M(d,d) \mid W = P_\pi W P_\pi^\top \text{ for all } (\pi,\pi) \in \Delta(S_{d-p} \times S_p)\}.$$

The gradient expressions corresponding to $\mathcal{W}_d$ are given in the body of the paper in Section 4.1.

In all expressions below:

- $\alpha_{(i)}^{(j)}$ (resp. $\beta_{(i)}^{(j)}$) denotes the angles between the ith row of W and the jth of W (resp. V, the target weight matrix).
- $\nu_{(i)}$ (resp. $\mu_{(i)}$) denotes the norm of the ith row of W (resp. V).
- $\nu_{(i)}^{(j)}$ (resp. $\mu_{(i)}^{(j)}$) denotes $\sin\arccos(\alpha_{(i)}^{(j)})$ (resp. $\sin\arccos(\beta_{(i)}^{(j)})$).

In Section F, we list the coefficients of the families of minima considered in Theorem 1 to $O(d^{-5/2})$-order.

C.1 Gradient expressions for $\mathcal{W}_1$

The space $\mathcal{W}_1$ is five-dimensional. A weight matrix for $d = 8$ can be parameterized as follows

$$\begin{bmatrix} a_1 & a_2 & a_2 & a_2 & a_2 & a_2 & a_2 & a_3 \\ a_2 & a_1 & a_2 & a_2 & a_2 & a_2 & a_2 & a_3 \\ a_2 & a_2 & a_1 & a_2 & a_2 & a_2 & a_2 & a_3 \\ a_2 & a_2 & a_2 & a_1 & a_2 & a_2 & a_2 & a_3 \\ a_2 & a_2 & a_2 & a_2 & a_1 & a_2 & a_2 & a_3 \\ a_2 & a_2 & a_2 & a_2 & a_2 & a_1 & a_2 & a_3 \\ a_2 & a_2 & a_2 & a_2 & a_2 & a_2 & a_1 & a_3 \\ a_4 & a_4 & a_4 & a_4 & a_4 & a_4 & a_4 & a_5 \end{bmatrix}.$$

The gradient entries, denoted by g_1, g_2, g_3, g_4 and g_5, are:

$$\begin{aligned} g_1 =& \frac{a_1(d-1)}{2} + \frac{a_1\left(d^2-3d+2\right)\sin\left(\alpha_{(1)}^{(2)}\right)}{2\pi} + \frac{a_1\nu_{(d)}(d-1)\sin\left(\alpha_{(1)}^{(d)}\right)}{2\pi\nu_{(1)}} \\ &- \frac{a_1(d-1)\sin\left(\beta_{(1)}^{(1)}\right)}{2\pi\nu_{(1)}} - \frac{a_1(d-1)\sin\left(\beta_{(1)}^{(d)}\right)}{2\pi\nu_{(1)}} \\ &- \frac{a_1\left(d^2-3d+2\right)\sin\left(\beta_{(1)}^{(2)}\right)}{2\pi\nu_{(1)}} - \frac{a_2\alpha_{(1)}^{(2)}\left(d^2-3d+2\right)}{2\pi} \\ &+ \frac{a_2\left(d^2-3d+2\right)}{2} - \frac{a_4\alpha_{(1)}^{(d)}(d-1)}{2\pi} + \frac{a_4(d-1)}{2} + \frac{\beta_{(1)}^{(1)}(d-1)}{2\pi} - \frac{d}{2} + \frac{1}{2}, \\ g_2 =& -\frac{a_1\alpha_{(1)}^{(2)}\left(d^2-3d+2\right)}{2\pi} + \frac{a_1\left(d^2-3d+2\right)}{2} - \frac{a_2\alpha_{(1)}^{(2)}\left(d^3-6d^2+11d-6\right)}{2\pi} \\ &+ \frac{a_2\left(d^3-5d^2+8d-4\right)\sin\left(\alpha_{(1)}^{(2)}\right)}{2\pi} + \frac{a_2\left(d^3-5d^2+8d-4\right)}{2} \\ &+ \frac{a_2\nu_{(d)}\left(d^2-3d+2\right)\sin\left(\alpha_{(1)}^{(d)}\right)}{2\pi\nu_{(1)}} - \frac{a_2\left(d^2-3d+2\right)\sin\left(\beta_{(1)}^{(1)}\right)}{2\pi\nu_{(1)}} \\ &- \frac{a_2\left(d^2-3d+2\right)\sin\left(\beta_{(1)}^{(d)}\right)}{2\pi\nu_{(1)}} - \frac{a_2\left(d^3-5d^2+8d-4\right)\sin\left(\beta_{(1)}^{(2)}\right)}{2\pi\nu_{(1)}} \\ &- \frac{a_4\alpha_{(1)}^{(d)}\left(d^2-3d+2\right)}{2\pi} + \frac{a_4\left(d^2-3d+2\right)}{2} + \frac{\beta_{(1)}^{(2)}\left(d^2-3d+2\right)}{2\pi} - \frac{d^2}{2} + \frac{3d}{2} - 1, \end{aligned}$$

$$
\begin{aligned}
g_3 =& -\frac{a_3\alpha_{(1)}^{(2)}\left(d^2-3d+2\right)}{2\pi}+\frac{a_3\left(d^2-3d+2\right)\sin\left(\alpha_{(1)}^{(2)}\right)}{2\pi}+\frac{a_3\left(d^2-2d+1\right)}{2} \\
&+\frac{a_3\nu_{(d)}\left(d-1\right)\sin\left(\alpha_{(1)}^{(d)}\right)}{2\pi\nu_{(1)}}-\frac{a_3\left(d-1\right)\sin\left(\beta_{(1)}^{(1)}\right)}{2\pi\nu_{(1)}}-\frac{a_3\left(d-1\right)\sin\left(\beta_{(1)}^{(d)}\right)}{2\pi\nu_{(1)}} \\
&-\frac{a_3\left(d^2-3d+2\right)\sin\left(\beta_{(1)}^{(2)}\right)}{2\pi\nu_{(1)}}-\frac{a_5\alpha_{(1)}^{(d)}\left(d-1\right)}{2\pi}+\frac{a_5\left(d-1\right)}{2}+\frac{\beta_{(1)}^{(d)}\left(d-1\right)}{2\pi}-\frac{d}{2}+\frac{1}{2}, \\
g_4 =& -\frac{a_1\alpha_{(1)}^{(d)}\left(d-1\right)}{2\pi}+\frac{a_1\left(d-1\right)}{2}-\frac{a_2\alpha_{(1)}^{(d)}\left(d^2-3d+2\right)}{2\pi} \\
&+\frac{a_2\left(d^2-3d+2\right)}{2}+\frac{a_4\nu_{(1)}\left(d^2-2d+1\right)\sin\left(\alpha_{(1)}^{(d)}\right)}{2\pi\nu_{(d)}}+\frac{a_4\left(d-1\right)}{2} \\
&-\frac{a_4\left(d-1\right)\sin\left(\beta_{(d)}^{(d)}\right)}{2\pi\nu_{(d)}}-\frac{a_4\left(d^2-2d+1\right)\sin\left(\beta_{(d)}^{(1)}\right)}{2\pi\nu_{(d)}}+\frac{\beta_{(d)}^{(1)}\left(d-1\right)}{2\pi}-\frac{d}{2}+\frac{1}{2}, \\
g_5 =& -\frac{a_3\alpha_{(1)}^{(d)}\left(d-1\right)}{2\pi}+\frac{a_3\left(d-1\right)}{2}+\frac{a_5\nu_{(1)}\left(d-1\right)\sin\left(\alpha_{(1)}^{(d)}\right)}{2\pi\nu_{(d)}} \\
&+\frac{a_5}{2}-\frac{a_5\left(d-1\right)\sin\left(\beta_{(d)}^{(1)}\right)}{2\pi\nu_{(d)}}-\frac{a_5\sin\left(\beta_{(d)}^{(d)}\right)}{2\pi\nu_{(d)}}+\frac{\beta_{(d)}^{(d)}}{2\pi}-\frac{1}{2}.
\end{aligned}
$$

C.2 Gradient expressions for $\mathcal{W}_2$

The space $\mathcal{W}_2$ is six-dimensional. A weight matrix for $d = 8$ can be parameterized as follows

$$
\begin{bmatrix}
a_1 & a_2 & a_2 & a_2 & a_2 & a_2 & a_3 & a_3 \\
a_2 & a_1 & a_2 & a_2 & a_2 & a_2 & a_3 & a_3 \\
a_2 & a_2 & a_1 & a_2 & a_2 & a_2 & a_3 & a_3 \\
a_2 & a_2 & a_2 & a_1 & a_2 & a_2 & a_3 & a_3 \\
a_2 & a_2 & a_2 & a_2 & a_1 & a_2 & a_3 & a_3 \\
a_2 & a_2 & a_2 & a_2 & a_2 & a_1 & a_3 & a_3 \\
a_4 & a_4 & a_4 & a_4 & a_4 & a_4 & a_5 & a_6 \\
a_4 & a_4 & a_4 & a_4 & a_4 & a_4 & a_6 & a_5
\end{bmatrix}.
$$

The gradient entries, denoted by g_1, g_2, g_3, g_4, g_5 and g_6, are:

$$
\begin{aligned}
g_1 =& \frac{a_1\left(d-2\right)}{2}+\frac{a_1\left(d^2-5d+6\right)\sin\left(\alpha_{(1)}^{(2)}\right)}{2\pi}+\frac{a_1\nu_{(d-1)}\left(d-2\right)\sin\left(\alpha_{(1)}^{(d-1)}\right)}{\pi\nu_{(1)}} \\
&-\frac{a_1\left(d-2\right)\sin\left(\beta_{(1)}^{(1)}\right)}{2\pi\nu_{(1)}}-\frac{a_1\left(d-2\right)\sin\left(\beta_{(1)}^{(d-1)}\right)}{\pi\nu_{(1)}} \\
&-\frac{a_1\left(d^2-5d+6\right)\sin\left(\beta_{(1)}^{(2)}\right)}{2\pi\nu_{(1)}}-\frac{a_2\alpha_{(1)}^{(2)}\left(d^2-5d+6\right)}{2\pi} \\
&+\frac{a_2\left(d^2-5d+6\right)}{2}-\frac{a_4\alpha_{(1)}^{(d-1)}\left(d-2\right)}{\pi}+a_4\left(d-2\right)+\frac{\beta_{(1)}^{(1)}\left(d-2\right)}{2\pi}-\frac{d}{2}+1,
\end{aligned}
$$

$$
\begin{aligned}
g_2 \;=& -\frac{a_1\alpha_{(1)}^{(2)}\left(d^2-5d+6\right)}{2\pi}+\frac{a_1\left(d^2-5d+6\right)}{2}-\frac{a_2\alpha_{(1)}^{(2)}\left(d^3-9d^2+26d-24\right)}{2\pi}\\
&+\frac{a_2\left(d^3-8d^2+21d-18\right)\sin\left(\alpha_{(1)}^{(2)}\right)}{2\pi}+\frac{a_2\left(d^3-8d^2+21d-18\right)}{2}\\
&+\frac{a_2\nu_{(d-1)}\left(d^2-5d+6\right)\sin\left(\alpha_{(1)}^{(d-1)}\right)}{\pi\nu_{(1)}}-\frac{a_2\left(d^2-5d+6\right)\sin\left(\beta_{(1)}^{(1)}\right)}{2\pi\nu_{(1)}}\\
&-\frac{a_2\left(d^2-5d+6\right)\sin\left(\beta_{(1)}^{(d-1)}\right)}{\pi\nu_{(1)}}-\frac{a_2\left(d^3-8d^2+21d-18\right)\sin\left(\beta_{(1)}^{(2)}\right)}{2\pi\nu_{(1)}}\\
&-\frac{a_4\alpha_{(1)}^{(d-1)}\left(d^2-5d+6\right)}{\pi}+a_4\left(d^2-5d+6\right)+\frac{\beta_{(1)}^{(2)}\left(d^2-5d+6\right)}{2\pi}-\frac{d^2}{2}+\frac{5d}{2}-3,\\
g_3 \;=& -\frac{a_3\alpha_{(1)}^{(2)}\left(d^2-5d+6\right)}{\pi}+\frac{a_3\left(d^2-5d+6\right)\sin\left(\alpha_{(1)}^{(2)}\right)}{\pi}\\
&+a_3\left(d^2-4d+4\right)+\frac{2a_3\nu_{(d-1)}\left(d-2\right)\sin\left(\alpha_{(1)}^{(d-1)}\right)}{\pi\nu_{(1)}}-\frac{a_3\left(d-2\right)\sin\left(\beta_{(1)}^{(1)}\right)}{\pi\nu_{(1)}}\\
&-\frac{2a_3\left(d-2\right)\sin\left(\beta_{(1)}^{(d-1)}\right)}{\pi\nu_{(1)}}-\frac{a_3\left(d^2-5d+6\right)\sin\left(\beta_{(1)}^{(2)}\right)}{\pi\nu_{(1)}}-\frac{a_5\alpha_{(1)}^{(d-1)}\left(d-2\right)}{\pi}\\
&+a_5\left(d-2\right)-\frac{a_6\alpha_{(1)}^{(d-1)}\left(d-2\right)}{\pi}+a_6\left(d-2\right)+\frac{\beta_{(1)}^{(d-1)}\left(d-2\right)}{\pi}-d+2,
\end{aligned}
$$

$$
\begin{aligned}
g_4 \;=& -\frac{a_1\alpha_{(1)}^{(d-1)}\left(d-2\right)}{\pi}+a_1\left(d-2\right)-\frac{a_2\alpha_{(1)}^{(d-1)}\left(d^2-5d+6\right)}{\pi}\\
&+a_2\left(d^2-5d+6\right)-\frac{a_4\alpha_{(d-1)}^{(d)}\left(d-2\right)}{\pi}+\frac{a_4\nu_{(1)}\left(d^2-4d+4\right)\sin\left(\alpha_{(1)}^{(d-1)}\right)}{\pi\nu_{(d-1)}}\\
&+\frac{a_4\left(d-2\right)\sin\left(\alpha_{(d-1)}^{(d)}\right)}{\pi}+2a_4\left(d-2\right)-\frac{a_4\left(d-2\right)\sin\left(\beta_{(d-1)}^{(d)}\right)}{\pi\nu_{(d-1)}}\\
&-\frac{a_4\left(d-2\right)\sin\left(\beta_{(d-1)}^{(d-1)}\right)}{\pi\nu_{(d-1)}}-\frac{a_4\left(d^2-4d+4\right)\sin\left(\beta_{(d-1)}^{(1)}\right)}{\pi\nu_{(d-1)}}+\frac{\beta_{(d-1)}^{(1)}\left(d-2\right)}{\pi}-d+2,\\
g_5 \;=& -\frac{a_3\alpha_{(1)}^{(d-1)}\left(d-2\right)}{\pi}+a_3\left(d-2\right)+\frac{a_5\nu_{(1)}\left(d-2\right)\sin\left(\alpha_{(1)}^{(d-1)}\right)}{\pi\nu_{(d-1)}}+\frac{a_5\sin\left(\alpha_{(d-1)}^{(d)}\right)}{\pi}+a_5\\
&-\frac{a_5\left(d-2\right)\sin\left(\beta_{(d-1)}^{(1)}\right)}{\pi\nu_{(d-1)}}-\frac{a_5\sin\left(\beta_{(d-1)}^{(d)}\right)}{\pi\nu_{(d-1)}}-\frac{a_5\sin\left(\beta_{(d-1)}^{(d-1)}\right)}{\pi\nu_{(d-1)}}-\frac{a_6\alpha_{(d-1)}^{(d)}}{\pi}+a_6+\frac{\beta_{(d-1)}^{(d-1)}}{\pi}-1\\
g_6 \;=& -\frac{a_3\alpha_{(1)}^{(d-1)}\left(d-2\right)}{\pi}+a_3\left(d-2\right)-\frac{a_5\alpha_{(d-1)}^{(d)}}{\pi}+a_5\\
&+\frac{a_6\nu_{(1)}\left(d-2\right)\sin\left(\alpha_{(1)}^{(d-1)}\right)}{\pi\nu_{(d-1)}}+\frac{a_6\sin\left(\alpha_{(d-1)}^{(d)}\right)}{\pi}+a_6\\
&-\frac{a_6\left(d-2\right)\sin\left(\beta_{(d-1)}^{(1)}\right)}{\pi\nu_{(d-1)}}-\frac{a_6\sin\left(\beta_{(d-1)}^{(d)}\right)}{\pi\nu_{(d-1)}}-\frac{a_6\sin\left(\beta_{(d-1)}^{(d-1)}\right)}{\pi\nu_{(d-1)}}+\frac{\beta_{(d-1)}^{(d)}}{\pi}-1.
\end{aligned}
$$

C.3 Gradient expressions for $\mathcal{W}_3$

The space $\mathcal{W}_3$ is six-dimensional. A weight matrix for $d = 8$ can be parameterized as follows

$$\begin{bmatrix} a_1 & a_2 & a_2 & a_2 & a_2 & a_3 & a_3 & a_3 \\ a_2 & a_1 & a_2 & a_2 & a_2 & a_3 & a_3 & a_3 \\ a_2 & a_2 & a_1 & a_2 & a_2 & a_3 & a_3 & a_3 \\ a_2 & a_2 & a_2 & a_1 & a_2 & a_3 & a_3 & a_3 \\ a_2 & a_2 & a_2 & a_2 & a_1 & a_3 & a_3 & a_3 \\ a_4 & a_4 & a_4 & a_4 & a_4 & a_5 & a_6 & a_6 \\ a_4 & a_4 & a_4 & a_4 & a_4 & a_6 & a_5 & a_6 \\ a_4 & a_4 & a_4 & a_4 & a_4 & a_6 & a_6 & a_5 \end{bmatrix}$$

The gradient entries, denoted by g_1, g_2, g_3, g_4, g_5 and g_6, are:

$$\begin{aligned} g_1 =& \frac{a_1\left(d-3\right)}{2} + \frac{a_1\left(d^2-7d+12\right)\sin\left(\alpha^{(2)}_{(1)}\right)}{2\pi} + \frac{3a_1\nu_{(d-2)}\left(d-3\right)\sin\left(\alpha^{(d-2)}_{(1)}\right)}{2\pi\nu_{(1)}} \\ &- \frac{a_1\left(d-3\right)\sin\left(\beta^{(1)}_{(1)}\right)}{2\pi\nu_{(1)}} - \frac{3a_1\left(d-3\right)\sin\left(\beta^{(d-2)}_{(1)}\right)}{2\pi\nu_{(1)}} \\ &- \frac{a_1\left(d^2-7d+12\right)\sin\left(\beta^{(2)}_{(1)}\right)}{2\pi\nu_{(1)}} - \frac{a_2\alpha^{(2)}_{(1)}\left(d^2-7d+12\right)}{2\pi} + \frac{a_2\left(d^2-7d+12\right)}{2} \\ &- \frac{3a_4\alpha^{(d-2)}_{(1)}\left(d-3\right)}{2\pi} + \frac{3a_4\left(d-3\right)}{2} + \frac{\beta^{(1)}_{(1)}\left(d-3\right)}{2\pi} - \frac{d}{2} + \frac{3}{2}, \\ g_2 =& -\frac{a_1\alpha^{(2)}_{(1)}\left(d^2-7d+12\right)}{2\pi} + \frac{a_1\left(d^2-7d+12\right)}{2} - \frac{a_2\alpha^{(2)}_{(1)}\left(d^3-12d^2+47d-60\right)}{2\pi} \\ &+ \frac{a_2\left(d^3-11d^2+40d-48\right)\sin\left(\alpha^{(2)}_{(1)}\right)}{2\pi} + \frac{a_2\left(d^3-11d^2+40d-48\right)}{2} \\ &+ \frac{3a_2\nu_{(d-2)}\left(d^2-7d+12\right)\sin\left(\alpha^{(d-2)}_{(1)}\right)}{2\pi\nu_{(1)}} - \frac{a_2\left(d^2-7d+12\right)\sin\left(\beta^{(1)}_{(1)}\right)}{2\pi\nu_{(1)}} \\ &- \frac{3a_2\left(d^2-7d+12\right)\sin\left(\beta^{(d-2)}_{(1)}\right)}{2\pi\nu_{(1)}} - \frac{a_2\left(d^3-11d^2+40d-48\right)\sin\left(\beta^{(2)}_{(1)}\right)}{2\pi\nu_{(1)}} \\ &- \frac{3a_4\alpha^{(d-2)}_{(1)}\left(d^2-7d+12\right)}{2\pi} + \frac{3a_4\left(d^2-7d+12\right)}{2} + \frac{\beta^{(2)}_{(1)}\left(d^2-7d+12\right)}{2\pi} - \frac{d^2}{2} + \frac{7d}{2} - 6, \\ g_3 =& -\frac{3a_3\alpha^{(2)}_{(1)}\left(d^2-7d+12\right)}{2\pi} + \frac{3a_3\left(d^2-7d+12\right)\sin\left(\alpha^{(2)}_{(1)}\right)}{2\pi} + \frac{3a_3\left(d^2-6d+9\right)}{2} \\ &+ \frac{9a_3\nu_{(d-2)}\left(d-3\right)\sin\left(\alpha^{(d-2)}_{(1)}\right)}{2\pi\nu_{(1)}} - \frac{3a_3\left(d-3\right)\sin\left(\beta^{(1)}_{(1)}\right)}{2\pi\nu_{(1)}} - \frac{9a_3\left(d-3\right)\sin\left(\beta^{(d-2)}_{(1)}\right)}{2\pi\nu_{(1)}} \\ &- \frac{3a_3\left(d^2-7d+12\right)\sin\left(\beta^{(2)}_{(1)}\right)}{2\pi\nu_{(1)}} - \frac{3a_5\alpha^{(d-2)}_{(1)}\left(d-3\right)}{2\pi} + \frac{3a_5\left(d-3\right)}{2} - \frac{3a_6\alpha^{(d-2)}_{(1)}\left(d-3\right)}{\pi} \\ &+ 3a_6\left(d-3\right) + \frac{3\beta^{(d-2)}_{(1)}\left(d-3\right)}{2\pi} - \frac{3d}{2} + \frac{9}{2}, \end{aligned}$$

$$
\begin{aligned}
g_4 \;=& -\frac{3a_1\alpha^{(d-2)}_{(1)}\left(d-3\right)}{2\pi}+\frac{3a_1\left(d-3\right)}{2}-\frac{3a_2\alpha^{(d-2)}_{(1)}\left(d^2-7d+12\right)}{2\pi}\\
&+\frac{3a_2\left(d^2-7d+12\right)}{2}-\frac{3a_4\alpha^{(d-1)}_{(d-2)}\left(d-3\right)}{\pi}+\frac{3a_4\nu_{(1)}\left(d^2-6d+9\right)\sin\left(\alpha^{(d-2)}_{(1)}\right)}{2\pi\nu_{(d-2)}}\\
&+\frac{3a_4\left(d-3\right)\sin\left(\alpha^{(d-1)}_{(d-2)}\right)}{\pi}+\frac{9a_4\left(d-3\right)}{2}-\frac{3a_4\left(d-3\right)\sin\left(\beta^{(d-1)}_{(d-2)}\right)}{\pi\nu_{(d-2)}}\\
&-\frac{3a_4\left(d-3\right)\sin\left(\beta^{(d-2)}_{(d-2)}\right)}{2\pi\nu_{(d-2)}}-\frac{3a_4\left(d^2-6d+9\right)\sin\left(\beta^{(1)}_{(d-2)}\right)}{2\pi\nu_{(d-2)}}+\frac{3\beta^{(1)}_{(d-2)}\left(d-3\right)}{2\pi}-\frac{3d}{2}+\frac{9}{2},\\
g_5 \;=& -\frac{3a_3\alpha^{(d-2)}_{(1)}\left(d-3\right)}{2\pi}+\frac{3a_3\left(d-3\right)}{2}+\frac{3a_5\nu_{(1)}\left(d-3\right)\sin\left(\alpha^{(d-2)}_{(1)}\right)}{2\pi\nu_{(d-2)}}\\
&+\frac{3a_5\sin\left(\alpha^{(d-1)}_{(d-2)}\right)}{\pi}+\frac{3a_5}{2}-\frac{3a_5\left(d-3\right)\sin\left(\beta^{(1)}_{(d-2)}\right)}{2\pi\nu_{(d-2)}}-\frac{3a_5\sin\left(\beta^{(d-1)}_{(d-2)}\right)}{\pi\nu_{(d-2)}}\\
&-\frac{3a_5\sin\left(\beta^{(d-2)}_{(d-2)}\right)}{2\pi\nu_{(d-2)}}-\frac{3a_6\alpha^{(d-1)}_{(d-2)}}{\pi}+3a_6+\frac{3\beta^{(d-2)}_{(d-2)}}{2\pi}-\frac{3}{2},\\
g_6 \;=& -\frac{3a_3\alpha^{(d-2)}_{(1)}\left(d-3\right)}{\pi}+3a_3\left(d-3\right)-\frac{3a_5\alpha^{(d-1)}_{(d-2)}}{\pi}+3a_5-\frac{3a_6\alpha^{(d-1)}_{(d-2)}}{\pi}\\
&+\frac{3a_6\nu_{(1)}\left(d-3\right)\sin\left(\alpha^{(d-2)}_{(1)}\right)}{\pi\nu_{(d-2)}}+\frac{6a_6\sin\left(\alpha^{(d-1)}_{(d-2)}\right)}{\pi}+6a_6\\
&-\frac{3a_6\left(d-3\right)\sin\left(\beta^{(1)}_{(d-2)}\right)}{\pi\nu_{(d-2)}}-\frac{6a_6\sin\left(\beta^{(d-1)}_{(d-2)}\right)}{\pi\nu_{(d-2)}}-\frac{3a_6\sin\left(\beta^{(d-2)}_{(d-2)}\right)}{\pi\nu_{(d-2)}}+\frac{3\beta^{(d-1)}_{(d-2)}}{\pi}-3.
\end{aligned}
$$

D Hessian spectrum

Below, we describe the technique we use to derive an analytic description of the Hessian spectrum. Some parts follow [4] verbatim. In order to avoid a long preliminaries section, key ideas and concepts are introduced and organized so as to illuminate our strategy for analyzing the Hessian. We illustrate with reference to the global minimum $W = V$ where $d = k$, the second layer is all ones, and the target weight matrix V is the identity I_d. In Section E, we provide the eigenvalue expressions for $\Delta(S_{d-1} \times S_1)$, organized by their isotypic component.

D.1 Studying invariance properties via group action

We first review background material on group actions and fix notations (see [20, Chapters 1, 2] for a more complete account). Elementary concepts from group theory are assumed known. We start with two examples that are used later.

Examples 1. (1) The *symmetric group* S_d, $d \in \mathbb{N}$, is the group of permutations of $[d] \doteq \{1, \ldots, d\}$.
(2) Let $\mathrm{GL}(d, \mathbb{R})$ denote the space of invertible linear maps on $\mathbb{R}^d$. Under composition, $\mathrm{GL}(d, \mathbb{R})$ has the structure of a group. The *orthogonal group* $\mathrm{O}(d)$ is the subgroup of $\mathrm{GL}(d, \mathbb{R})$ defined by $\mathrm{O}(d) = \{A \in \mathrm{GL}(d, \mathbb{R}) \mid \|Ax\| = \|x\|, \text{ for all } x \in \mathbb{R}^d\}$. Both $\mathrm{GL}(d, \mathbb{R})$ and $\mathrm{O}(d)$ can be viewed as groups of invertible $d \times d$ matrices.

Characteristically, these groups consist of *transformations* of a set and so we are led to the notion of a *G-space* X where we have an *action* of a group G on a set X. Formally, this is a group homomorphism from G to the group of bijections of X. For example, S_d naturally acts on $[d]$ as permutations and both $\mathrm{GL}(d, \mathbb{R})$ and $\mathrm{O}(d)$ act on $\mathbb{R}^d$ as linear transformations (or matrix multiplication).

An example, which we use extensively in studying the invariance properties of $\mathcal{L}$, is given by the action of the group $S_k \times S_d \subset S_{k\times d},\ k, d \in \mathbb{N}$, on $[k] \times [d]$ defined by

$$(\pi, \rho)(i, j) = (\pi^{-1}(i), \rho^{-1}(j)),\ \pi \in S_k, \rho \in S_d,\ (i, j) \in [k] \times [d]. \tag{14}$$

This action induces an action on the space $M(k, d)$ of $k \times d$-matrices $A = [A_{ij}]$ by $(\pi, \rho)[A_{ij}] = [A_{\pi^{-1}(i),\rho^{-1}(j)}]$. The action can be defined in terms of permutation matrices but is easier to describe in terms of rows and columns: $(\pi, \rho)A$ permutes rows (resp. columns) of A according to π (resp. ρ). As mentioned in the introduction, for our choice of $V = I_d$, $\mathcal{L}$ is $S_k \times S_d$-invariant. Note that $\Delta S_d \approx S_d$. When we restrict the $S_d \times S_d$-action on $M(k, k)$ to ΔS_d, we refer to the diagonal S_d-action, or just the S_d-action on $M(d, d)$. This action of S_d on $M(d, d)$ maps diagonal matrices to diagonal matrices and should not be confused with the actions of S_d on $M(d, d)$ defined by either permuting rows or columns.

Example 2. Take $p, q \in \mathbb{N}, p+q = d$, and consider the diagonal action of $S_p \times S_q \subset S_d$ on $M(d, d)$. Write $A \in M(d, d)$ in block matrix form as $A = \begin{bmatrix} A_{p,p} & A_{p,q} \\ A_{q,p} & A_{q,q} \end{bmatrix}$. If $(g, h) \in S_p \times S_q \subset S_d$, then $(g, h)A = \begin{bmatrix} gA_{p,p} & (g, h)A_{p,q} \\ (h, g)A_{q,p} & hA_{q,q} \end{bmatrix}$ where $gA_{p,p}$ (resp. $hA_{q,q}$) are defined via the diagonal action of S_p (resp. S_q) on $A_{p,p}$ (resp. $A_{q,q}$), and $(g, h)A_{p,q}$ and $(h, g)A_{q,p}$ are defined through the natural action of $S_p \times S_q$ on rows and columns. Thus, for $(g, h)A_{p,q}$ (resp. $(h, g)A_{q,p}$) we permute rows (resp. columns) according to g and columns (resp. rows) according to h. In the case when $p = d - 1$, $q = 1$, S_{d-1} will act diagonally on $A_{d-1,d-1}$, fix a_{dd}, and act by permuting the first $(d - 1)$ entries of the last row and column.

As mentioned in body of the paper, given $W \in M(d, d)$, the largest subgroup of $S_d \times S_d$ fixing W is called the *isotropy* subgroup of W and is used as means of measuring the symmetry of W. The isotropy subgroup of $V \in M(d, d)$ is the diagonal subgroup ΔS_d. Our focus will be on critical points W whose isotropy groups are subgroups of the target matrix $V = I_d$, that is, ΔS_d and $\Delta(S_{d-1} \times S_1)$ (see Figure 2—we use the notation ΔS_d as the isotropy is a *subgroup* of $S_d \times S_d$). In the next section, we show how the symmetry of local minima greatly simplifies the analysis of their Hessian.

D.2 The spectrum of equivariant linear isomorphisms

If G is a subgroup of $\mathrm{O}(d)$, the action on $\mathbb{R}^d$ is called an *orthogonal* representation of G (we often drop the qualifier orthogonal). Denote by $(\mathbb{R}^d, G)$ as necessary. The *degree* of a representation (V, G) is the dimension of V (V will always be a linear subspace of some $\mathbb{R}^n$ with the induced Euclidean inner product). The action of $S_k \times S_d \subset S_{k\times d}$ on $M(k, d)$ is orthogonal with respect to the standard Euclidean inner product on $M(k, d) \approx \mathbb{R}^{k\times d}$ since the action permutes the coordinates of $\mathbb{R}^{k\times d}$ (equivalently, components of $k \times d$ matrices). Given two representations (V, G) and (W, G), a map $A : V \to W$ is called G-equivariant if $A(gv) = gA(v)$, for all $g \in G, v \in V$. If A is linear and equivariant, we say A is a *G-map*. Invariant functions naturally provide examples of equivariant maps. Thus the gradient $\nabla\mathcal{L}$ is a $S_k \times S_d$-equivariant self map of $M(k, d)$ and if W is a critical point of $\nabla\mathcal{L}$ with isotropy $G \subset S_k \times S_d$, then $\nabla^2\mathcal{L}(W) : M(k, d) \to M(k, d)$ is a G-map (see [6]). The equivariance of the Hessian is the key ingredient that allows us to study the spectral density at *symmetric* local minima.

A representation $(\mathbb{R}^n, G)$ is *irreducible* if the only linear subspaces of $\mathbb{R}^n$ that are preserved (invariant) by the G-action are $\mathbb{R}^n$ and $\{0\}$. Two orthogonal representations (V, G), (W, G) are *isomorphic* (and have the same *isomorphism class*) if there exists a G-map $A : V \to W$ which is a linear isomorphism. If (V, G), (W, G) are irreducible but not isomorphic then every G-map $A : V \to W$ is zero (as the kernel and the image of a G-map are G-invariant). If (V, G) is irreducible, then the space $\mathrm{Hom}_G(V, V)$ of G-maps (endomorphisms) of V is a real associative division algebra and is isomorphic by a theorem of Frobenius to either $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ (the quaternions). The *only* case that will concern us here is when $\mathrm{Hom}_G(V, V) \approx \mathbb{R}$ when we say the representation is *real*.

Example 3. Let $n > 1$. Take the natural (orthogonal) action of S_n on $\mathbb{R}^n$ defined by permuting coordinates. The representation is not irreducible since the subspace $T = \{(x, x, \cdots, x) \in \mathbb{R}^n \mid x \in \mathbb{R}\}$ is invariant by the action of S_n, as is the hyperplane $H_{n-1} = T^\perp = \{(x_1, \cdots, x_n) \mid \sum_{i\in[n]} x_i = 0\}$. It is easy to check that (T, S_n), also called the *trivial* representation of S_n, and (H_{n-1}, S_n), the *standard* representation, are irreducible, real, and not isomorphic.

Every representation $(\mathbb{R}^n, G)$ can be written uniquely, up to order, as an orthogonal direct sum $\oplus_{i\in[m]} V_i$, where each (V_i, G) is an orthogonal direct sum of isomorphic irreducible representations (V_{ij}, G), $j \in [p_i]$, and (V_{ij}, G) is isomorphic to $(V_{i'j'}, G)$ if and only if $i' = i$. The subspaces V_{ij} are *not* uniquely determined if $p_i > 1$. If there are m distinct isomorphism classes $\mathfrak{v}_1, \cdots, \mathfrak{v}_m$ of irreducible representations, then $(\mathbb{R}^n, G)$ may be represented by the sum $p_1\mathfrak{v}_1 + \cdots + p_m\mathfrak{v}_m$, where $p_i \geq 1$ counts the number of representations with isomorphism class $\mathfrak{v}_i$. Up to order, this sum (that is, the $\mathfrak{v}_i$ and their multiplicities) is uniquely determined by $(\mathbb{R}^n, G)$. This is the *isotypic decomposition* of $(\mathbb{R}^n, G)$ (see [60]). The isotypic decomposition is a powerful tool for extracting information about the spectrum of G-maps.

If $G = S_d$, then every irreducible representation of S_d is real [22, Thm. 4.3]. Suppose, as above, that $(\mathbb{R}^n, S_d) = \oplus_{i\in[m]} V_i$ and $A : \mathbb{R}^n \to \mathbb{R}^n$ is an S_d-map. Since the induced maps $A_{ii'} : V_i \to V_{i'}$ must be zero if $i \neq i'$, A is uniquely determined by the S_d-maps $A_{ii} : V_i \to V_i$, $i \in [m]$. Fix i and choose an S_d-representation (W, S_d) in the isomorphism class $\mathfrak{v}_i$. Choose S_d-isomorphisms $W \to V_{ij}$, $j \in [p_i]$. Then A_{ii} induces $\overline{A}_{ii} : W^{p_i} \to W^{p_i}$ and so determines a (real) matrix $M_i \in M(p_i, p_i)$ since $\mathrm{Hom}_{S_d}(W, W) \approx \mathbb{R}$. Different choices of V_{ij}, or isomorphism $W \to V_{ij}$, yield a matrix similar to M_i. Each eigenvalue of M_i of multiplicity r gives an eigenvalue of A_{ii}, and so of A, of multiplicity $r\,\mathrm{degree}(\mathfrak{v}_i)$.

Fact 1. (Notations and assumptions as above.) If A is the Hessian, all eigenvalues are real and each eigenvalue of M_i of multiplicity r will be an eigenvalue of A with multiplicity $r\,\mathrm{degree}(\mathfrak{v}_i)$. In particular, A has most $\sum_{i\in[m]} p_i$ distinct real eigenvalues—regardless of the dimension of the underlying space.

Our strategy can be now summarized as follows. Given a local minima W, we compute the isotropy group $G \subset S_k \times S_d$ of W. Since the Hessian of $\mathcal{F}$ at W is a G-map, may use the isotypic decomposition of the action of G on $M(k, d)$ to extract the spectral properties of the Hessian. In our setting, local minima have large isotropy groups, typically, as large as $\Delta(S_p \times S_{d-p}),\ 0 \leq p < d/2$. Studying the Hessian at these minima requires the isotopic decomposition corresponding to $\Delta(S_p \times S_{d-p}),\ 0 \leq p < d/2$, which we detail in Theorem D.3 below.

D.3 The isotypic decomposition of $(M(d, d), S_d)$ and the spectrum at $W = V$

Regard $M(d, d)$ as an S_d-space (diagonal action). The trivial representation, denoted by $\mathfrak{t}_d$, and the standard representation, denoted by $\mathfrak{s}_d$, introduced in Example 3 are examples of the many irreducible representations of S_d. In the general theory, each irreducible representation of S_d is associated to a partition of the set $[d]$. The description of the isotypic decomposition of $(M(d, d), S_d)$ is relatively simple and uses just 4 irreducible representations of S_d for $d \geq 4$.

- The trivial representation $\mathfrak{t}_d$ of degree 1.
- The standard representation $\mathfrak{s}_d$ of S_d of degree $k - 1$.
- The exterior square representation $\mathfrak{x}_d = \wedge^2\mathfrak{s}_d$ of degree $\frac{(d-1)(d-2)}{2}$.
- A representation $\mathfrak{y}_d$ of degree $\frac{d(d-3)}{2}$. We describe $\mathfrak{y}_d$ explicitly later in terms of symmetric matrices (formally, it is the representation associated to the partition $(d-2, 2)$).

We omit the subscript d when clear from the context. Assume that $d \geq 4$. We begin with a well-known result about the representation $\mathfrak{s} \otimes \mathfrak{s}$ (see, e.g., [22]). If $\mathfrak{s} \odot \mathfrak{s}$ denotes the symmetric tensor product of $\mathfrak{s}$, then

$$\mathfrak{s} \otimes \mathfrak{s} = \mathfrak{s} \odot \mathfrak{s} + \mathfrak{x} = \mathfrak{t} + \mathfrak{s} + \mathfrak{y} + \mathfrak{x}. \tag{15}$$

Since all the irreducible S_d-representations are real, they are isomorphic to their dual representations and so we have the isotypic decomposition

$$M(k, k) \quad \approx \quad \mathbb{R}^k \otimes \mathbb{R}^k \approx (\mathfrak{s} + \mathfrak{t}) \otimes (\mathfrak{s} + \mathfrak{t}) = 2\mathfrak{t} + 3\mathfrak{s} + \mathfrak{x} + \mathfrak{y}, \tag{16}$$

since $\mathfrak{t} \otimes \mathfrak{s} = \mathfrak{s}$ and $\mathfrak{t} \otimes \mathfrak{t} = \mathfrak{t}$.

Using Fact 1, information can immediately be deduced from Equation (16). For example, if W is a critical point of isotropy ΔS_d (a fixed point of the S_d-action on $M(d, d)$), then the spectrum of the Hessian contains at most $2 + 3 + 1 + 1 = 7$ distinct eigenvalues which distribute as follows: $\mathfrak{t}$ contributes 2 eigenvalues of multiplicity 1, $\mathfrak{s}$ contributes 2 eigenvalues of multiplicity $d - 1$, $\mathfrak{x}$

contributes one eigenvalue of multiplicity $\frac{(d-1)(d-2)}{2}$, and $\mathfrak{y}$ contributes one eigenvalue of multiplicity $\frac{d(d-3)}{2}$. This applies to the global minimum $W = V$ and the spurious minimum of type A.

Next, we would like to compute the actual eigenvalues. We demonstrate the method for the single $\mathfrak{x}$-eigenvalue (the example given in the body of the paper in section Section 4.2 refers to the single $\mathfrak{y}$-eigenvalue). Pick a non-zero vector from the $\mathfrak{x}$-representation. For example,

$$\mathfrak{X}^d = \begin{bmatrix} 0 & 1 & \dots & 1 & -(d-2) \\ -1 & 0 & \dots & 0 & 1 \\ \dots & \dots & \dots & \dots & \dots \\ -1 & 0 & \dots & 0 & 1 \\ (d-2) & -1 & \dots & -1 & 0 \end{bmatrix},$$

where rows and columns sum to zero and the only non-zero entries are in rows and columns 1 and d. Let $\overline{\mathfrak{X}^d} \in \mathbb{R}^{d\times d}$ be defined by concatenating the rows of $\mathfrak{X}^d$. Since $\mathfrak{x}$ only occurs once in the isotopic decomposition and $\nabla^2\mathcal{L}(V)$ is S_d-equivariant, $\overline{\mathfrak{X}^d}$ must be an eigenvector. In particular, $(\nabla^2\mathcal{L}(V)\overline{\mathfrak{X}^d})_i = \lambda_{\mathfrak{x}}\overline{\mathfrak{X}^d_i}$, all $i \in [d^2]$. Choose i so that $\overline{\mathfrak{X}^d_i} \neq 0$. For example, $\overline{\mathfrak{X}^d_2} = 1$. Matrix multiplication, yields $\lambda_{\mathfrak{x}} = 1/4 - 1/2\pi$. A similar analysis holds for the eigenvalue associated to $\mathfrak{y}$. The multiple factors $2\mathfrak{t}$ and $3\mathfrak{s}$ are handled by making judicious choices of orthogonal invariant subspaces and representative vectors in $M(k,k)$.

Having described the general strategy for analyzing the Hessian spectrum for global minima, we now examine the spectrum at various types of spurious minima. We need two additional ingredients: a specification of the entries of a given family of spurious minima and the respective isotypic decomposition; we begin with the latter.

As discussed in the body of the paper, the symmetry-based analysis of the Hessian relies on the fact that isotropy groups of spurious minima tend to be (and some provably are) maximal subgroups of the target matrix isotropy. For $V = I$, the relevant maximal isotropy groups are of the form $\Delta(S_p \times S_q),\ p+q = d$. Below, we provide the corresponding isotypic decomposition. Assume $d = k$ and regard $M(d,d)$ as an $S_p \times S_q$-space, where $S_p \times S_q \subset S_d$ and the (diagonal) action of S_d is restricted to the subgroup $S_p \times S_q$.

Theorem ([4, Theorem 4])**.** *The isotypic decomposition of $(M(d,d), S_p \times S_q)$ is given by:*

1. *If $p = d-1$, $q = 1$, and $k \geq 5$,*

$$M(d,d) = 5\mathfrak{t} + 5\mathfrak{s}_{d-1} + \mathfrak{x}_{d-1} + \mathfrak{y}_{d-1}.$$

2. *If $q \geq 2$, $d-1 > p > p/2$ and $d \geq 4+q$, then*

$$M(d,d) = 6\mathfrak{t} + 5\mathfrak{s}_p + a\mathfrak{s}_q + \mathfrak{x}_p + \mathfrak{y}_p + b\mathfrak{x}_q + c\mathfrak{y}_q + 2\mathfrak{s}_p \boxtimes \mathfrak{s}_q,$$

where if $q = 2$, then $a = 4, b = c = 0$; if $q = 3$, then $a = 5, b = 1, c = 0$; and if $q \geq 4$, then $a = 5, b = c = 1$.

(There is a minor error in [4, Theorem 4]. The correct value for a in the second case is as stated here.)

In contrast to the setting considered in [4], in our case the second layer is trainable. To compute the isotypic decomposition corresponding to this case (i.e., $M(d,d) \times \mathbb{R}^d$), we simply treat the weights of the second layer as an additional row of the weight matrix of the first layer. The additional row is split into p entries and $d-p$ entries. This adds $\mathfrak{t} + \mathfrak{s}_p$ to the isotypic decomposition if $q = 0$, $2\mathfrak{t} + \mathfrak{s}_p$ to the isotypic decomposition if $q = 1$, and $2\mathfrak{t} + \mathfrak{s}_p + \mathfrak{s}_q$ otherwise.

Theorem 4. *The isotypic decomposition of $(M(d,d), S_p \times S_q) \otimes (\mathbb{R}^d, S_p \times S_q)$ is given by:*

1. *If $p = d$, $q = 0$, and $k \geq 5$,*

$$M(d,d) = 3\mathfrak{t} + 4\mathfrak{s}_d + \mathfrak{x}_d + \mathfrak{y}_d.$$

2. *If $p = d-1$, $q = 1$, and $k \geq 5$,*

$$M(d,d) = 7\mathfrak{t} + 6\mathfrak{s}_{d-1} + \mathfrak{x}_{d-1} + \mathfrak{y}_{d-1}.$$

3. *If $q \geq 2$, $d-1 > p > p/2$ and $d \geq 4+q$, then*

$$M(d,d) = 8\mathfrak{t} + 6\mathfrak{s}_p + a\mathfrak{s}_q + \mathfrak{x}_p + \mathfrak{y}_p + b\mathfrak{x}_q + c\mathfrak{y}_q + 2\mathfrak{s}_p \boxtimes \mathfrak{s}_q,$$

where if $q = 2$, then $a = 5, b = c = 0$; if $q = 3$, then $a = 6, b = 1, c = 0$; and if $q \geq 4$, then $a = 6, b = c = 1$.

Theorem 4 implies that the Hessian spectrum of local minima (or critical points) with isotropy $\Delta(S_p \times S_q)$ has at most 9 distinct eigenvalues if (1) applies, at most 15 distinct eigenvalues if (2) applies, and if (3) holds, at most 24 distinct eigenvalues if $q = 2$, at most 25 distinct eigenvalues if $q = 3$, and at most 26 distinct eigenvalues if $q \geq 4$. We omit some less interesting cases when k is small.

Following the same lines of argument described in Section D.3, the next step is to pick a set of non-zero representative vectors for each irreducible representation that will allow us to compute the spectrum. We adopt the same of choice of representative vectors from [4]. We demonstrate the final step with respect to the two trivial factors of S_d in (16). Let $\mathfrak{D}_1$ and $\mathfrak{D}_2$ be the two representatives and let $\nabla^2\mathcal{L}(\mathfrak{D}_i) = \alpha_{i1}\mathfrak{D}_1 + \alpha_{i2}\mathfrak{D}_2$, $i = 1, 2$. The eigenvalues of $\nabla^2\mathcal{L}|2\mathfrak{t}$ are then the eigenvalues of the 2×2 transition matrix $A = [\alpha_{ij}]$. To compute the eigenvalues of A to, say, $O(d^{-1/2})$-order, one solves the equation

$$\det(A - (x_1 d + x_2 d^{1/2} + x^3)) = 0,$$

for x_1, x_2, x_3, where sufficiently many coefficients of the power series of the entries of A are assumed known. The same recipe is used for computing the rest of the eigenvalues.

E Eigenvalues transition matrices

Below, we provide the explicit form of the eigenvalue transition matrices for the natural representation of S_d.

$\mathfrak{x}$-rep. Let the associated 1×1 transition matrix be denoted by $T^{\mathfrak{x}}$. Then,

$$\begin{aligned} T^{\mathfrak{x}}_{1,1} = &-\frac{a_1^2}{2\pi\nu_{(1)}^2\nu_{(1)}^{(2)}} + \frac{a_1^2\sin\left(\alpha_{(1)}^{(2)}\right)}{2\pi\nu_{(1)}^2\left(\nu_{(1)}^{(2)}\right)^2} + \frac{a_1a_2}{\pi\nu_{(1)}^2\nu_{(1)}^{(2)}} - \frac{a_1a_2\sin\left(\alpha_{(1)}^{(2)}\right)}{\pi\nu_{(1)}^2\left(\nu_{(1)}^{(2)}\right)^2} \\ &- \frac{a_2^2}{2\pi\nu_{(1)}^2\nu_{(1)}^{(2)}} + \frac{a_2^2\sin\left(\alpha_{(1)}^{(2)}\right)}{2\pi\nu_{(1)}^2\left(\nu_{(1)}^{(2)}\right)^2} + \frac{\alpha_{(1)}^{(2)}}{2\pi} + \frac{(d-1)\sin\left(\alpha_{(1)}^{(2)}\right)}{2\pi} \\ &- \frac{(d-1)\sin\left(\beta_{(1)}^{(2)}\right)}{2\pi\nu_{(1)}} - \frac{\sin\left(\beta_{(1)}^{(1)}\right)}{2\pi\nu_{(1)}} - \frac{\sin\left(\beta_{(1)}^{(2)}\right)}{2\pi\left(\mu_{(1)}^{(2)}\right)^2\nu_{(1)}}. \end{aligned}$$

$\mathfrak{y}$-rep. Let the associated 1×1 transition matrix be denoted by $T^{\mathfrak{y}}$. Then,

$$\begin{aligned} T^{\mathfrak{y}}_{1,1} = &\frac{a_1^2}{2\pi\nu_{(1)}^2\nu_{(1)}^{(2)}} + \frac{a_1^2\sin\left(\alpha_{(1)}^{(2)}\right)}{2\pi\nu_{(1)}^2\left(\nu_{(1)}^{(2)}\right)^2} - \frac{a_1a_2}{\pi\nu_{(1)}^2\nu_{(1)}^{(2)}} - \frac{a_1a_2\sin\left(\alpha_{(1)}^{(2)}\right)}{\pi\nu_{(1)}^2\left(\nu_{(1)}^{(2)}\right)^2} + \frac{a_2^2}{2\pi\nu_{(1)}^2\nu_{(1)}^{(2)}} + \frac{a_2^2\sin\left(\alpha_{(1)}^{(2)}\right)}{2\pi\nu_{(1)}^2\left(\nu_{(1)}^{(2)}\right)^2} \\ &+ \frac{\alpha_{(1)}^{(2)}}{2\pi} + \frac{(d-1)\sin\left(\alpha_{(1)}^{(2)}\right)}{2\pi} - \frac{(d-1)\sin\left(\beta_{(1)}^{(2)}\right)}{2\pi\nu_{(1)}} - \frac{\sin\left(\beta_{(1)}^{(1)}\right)}{2\pi\nu_{(1)}} - \frac{\sin\left(\beta_{(1)}^{(2)}\right)}{2\pi\left(\mu_{(1)}^{(2)}\right)^2\nu_{(1)}}. \end{aligned}$$

$\mathfrak{s}$-rep. Let the associated 4×4 transition matrix be denoted by $T^{\mathfrak{s}}$. Then,

$$T^{\mathfrak{s}}_{1,1} = -\frac{a_1^2 (d-1) \sin\left(\alpha^{(2)}_{(1)}\right)}{2\pi \nu^2_{(1)}} - \frac{a_1^2}{2\pi \nu^2_{(1)} \nu^{(2)}_{(1)}} + \frac{a_1^2 (d-1) \sin\left(\alpha^{(2)}_{(1)}\right) \cos^2\left(\alpha^{(2)}_{(1)}\right)}{2\pi \nu^2_{(1)} \left(\nu^{(2)}_{(1)}\right)^2}$$
$$+ \frac{a_1^2 (d-1) \sin\left(\beta^{(2)}_{(1)}\right)}{2\pi \nu^3_{(1)}} + \frac{a_1^2 \sin\left(\beta^{(1)}_{(1)}\right)}{2\pi \nu^3_{(1)}} - \frac{a_1^2 (d-1) \sin\left(\beta^{(2)}_{(1)}\right) \cos^2\left(\beta^{(2)}_{(1)}\right)}{2\pi \left(\mu^{(2)}_{(1)}\right)^2 \nu^3_{(1)}}$$
$$- \frac{a_1^2 \sin\left(\beta^{(1)}_{(1)}\right) \cos^2\left(\beta^{(1)}_{(1)}\right)}{2\pi \left(\mu^{(1)}_{(1)}\right)^2 \nu^3_{(1)}} + \frac{a_1 a_2 \cos\left(\alpha^{(2)}_{(1)}\right)}{\pi \nu^2_{(1)} \nu^{(2)}_{(1)}} - \frac{a_1 a_2 (d-1) \sin\left(\alpha^{(2)}_{(1)}\right) \cos\left(\alpha^{(2)}_{(1)}\right)}{\pi \nu^2_{(1)} \left(\nu^{(2)}_{(1)}\right)^2}$$
$$+ \frac{a_1 \sin\left(\beta^{(1)}_{(1)}\right) \cos\left(\beta^{(1)}_{(1)}\right)}{\pi \left(\mu^{(1)}_{(1)}\right)^2 \nu^2_{(1)}} - \frac{a_2^2}{2\pi \nu^2_{(1)} \nu^{(2)}_{(1)}} + \frac{a_2^2 (d-1) \sin\left(\alpha^{(2)}_{(1)}\right)}{2\pi \nu^2_{(1)} \left(\nu^{(2)}_{(1)}\right)^2}$$
$$+ \frac{(d-1) \sin\left(\alpha^{(2)}_{(1)}\right)}{2\pi} + \frac{1}{2} - \frac{(d-1) \sin\left(\beta^{(2)}_{(1)}\right)}{2\pi \nu_{(1)}} - \frac{\sin\left(\beta^{(1)}_{(1)}\right)}{2\pi \nu_{(1)}} - \frac{\sin\left(\beta^{(1)}_{(1)}\right)}{2\pi \left(\mu^{(1)}_{(1)}\right)^2 \nu_{(1)}},$$

$$T^{\mathfrak{s}}_{1,2} = \frac{a_1^2 d \cos\left(\alpha^{(2)}_{(1)}\right)}{2\pi \nu^2_{(1)} \nu^{(2)}_{(1)}} - \frac{a_1^2 d \sin\left(\alpha^{(2)}_{(1)}\right) \cos\left(\alpha^{(2)}_{(1)}\right)}{2\pi \nu^2_{(1)} \left(\nu^{(2)}_{(1)}\right)^2} - \frac{a_1 a_2 d}{\pi \nu^2_{(1)} \nu^{(2)}_{(1)}} + \frac{a_1 a_2 d \sin\left(\alpha^{(2)}_{(1)}\right)}{2\pi \nu^2_{(1)} \left(\nu^{(2)}_{(1)}\right)^2}$$
$$+ \frac{a_1 a_2 d \sin\left(\beta^{(1)}_{(1)}\right)}{2\pi \nu^3_{(1)}} - \frac{a_1 a_2 d \sin\left(\beta^{(1)}_{(1)}\right) \cos^2\left(\beta^{(1)}_{(1)}\right)}{2\pi \left(\mu^{(1)}_{(1)}\right)^2 \nu^3_{(1)}} - \frac{a_1 a_2 \left(d^2 - d\right) \sin\left(\alpha^{(2)}_{(1)}\right)}{2\pi \nu^2_{(1)}}$$
$$- \frac{a_1 a_2 \left(d^2 - 2d\right) \sin\left(\alpha^{(2)}_{(1)}\right) \cos\left(\alpha^{(2)}_{(1)}\right)}{2\pi \nu^2_{(1)} \left(\nu^{(2)}_{(1)}\right)^2} + \frac{a_1 a_2 \left(d^2 - d\right) \sin\left(\alpha^{(2)}_{(1)}\right) \cos^2\left(\alpha^{(2)}_{(1)}\right)}{2\pi \nu^2_{(1)} \left(\nu^{(2)}_{(1)}\right)^2}$$
$$+ \frac{a_1 a_2 \left(d^2 - d\right) \sin\left(\beta^{(2)}_{(1)}\right)}{2\pi \nu^3_{(1)}} - \frac{a_1 a_2 \left(d^2 - d\right) \sin\left(\beta^{(2)}_{(1)}\right) \cos^2\left(\beta^{(2)}_{(1)}\right)}{2\pi \left(\mu^{(2)}_{(1)}\right)^2 \nu^3_{(1)}}$$
$$+ \frac{a_1 d \sin\left(\beta^{(2)}_{(1)}\right) \cos\left(\beta^{(2)}_{(1)}\right)}{2\pi \left(\mu^{(2)}_{(1)}\right)^2 \nu^2_{(1)}} + \frac{a_2^2 d \cos\left(\alpha^{(2)}_{(1)}\right)}{2\pi \nu^2_{(1)} \nu^{(2)}_{(1)}} + \frac{a_2^2 \left(d^2 - 2d\right) \sin\left(\alpha^{(2)}_{(1)}\right)}{2\pi \nu^2_{(1)} \left(\nu^{(2)}_{(1)}\right)^2}$$
$$- \frac{a_2^2 \left(d^2 - d\right) \sin\left(\alpha^{(2)}_{(1)}\right) \cos\left(\alpha^{(2)}_{(1)}\right)}{2\pi \nu^2_{(1)} \left(\nu^{(2)}_{(1)}\right)^2} + \frac{a_2 d \sin\left(\beta^{(1)}_{(1)}\right) \cos\left(\beta^{(1)}_{(1)}\right)}{2\pi \left(\mu^{(1)}_{(1)}\right)^2 \nu^2_{(1)}} + \frac{\alpha^{(2)}_{(1)} d}{2\pi} - \frac{d}{2},$$

$$
\begin{aligned}
T^{\mathfrak{s}}_{1,3} = &-\frac{a_1^2\,(d-2)\cos\left(\alpha^{(2)}_{(1)}\right)}{2\pi\nu^2_{(1)}\nu^{(2)}_{(1)}} - \frac{a_1^2\,(d-2)\sin\left(\alpha^{(2)}_{(1)}\right)\cos\left(\alpha^{(2)}_{(1)}\right)}{2\pi\nu^2_{(1)}\left(\nu^{(2)}_{(1)}\right)^2} \\
&- \frac{a_1a_2\left(d^2-3d+2\right)\sin\left(\alpha^{(2)}_{(1)}\right)}{2\pi\nu^2_{(1)}} + \frac{a_1a_2\,(d-2)\cos\left(\alpha^{(2)}_{(1)}\right)}{\pi\nu^2_{(1)}\nu^{(2)}_{(1)}} + \frac{a_1a_2\,(d-2)\sin\left(\alpha^{(2)}_{(1)}\right)}{2\pi\nu^2_{(1)}\left(\nu^{(2)}_{(1)}\right)^2} \\
&- \frac{a_1a_2\left(d^2-4d+4\right)\sin\left(\alpha^{(2)}_{(1)}\right)\cos\left(\alpha^{(2)}_{(1)}\right)}{2\pi\nu^2_{(1)}\left(\nu^{(2)}_{(1)}\right)^2} + \frac{a_1a_2\left(d^2-3d+2\right)\sin\left(\alpha^{(2)}_{(1)}\right)\cos^2\left(\alpha^{(2)}_{(1)}\right)}{2\pi\nu^2_{(1)}\left(\nu^{(2)}_{(1)}\right)^2} \\
&+ \frac{a_1a_2\,(d-2)\sin\left(\beta^{(1)}_{(1)}\right)}{2\pi\nu^3_{(1)}} + \frac{a_1a_2\left(d^2-3d+2\right)\sin\left(\beta^{(2)}_{(1)}\right)}{2\pi\nu^3_{(1)}} \\
&- \frac{a_1a_2\left(d^2-3d+2\right)\sin\left(\beta^{(2)}_{(1)}\right)\cos^2\left(\beta^{(2)}_{(1)}\right)}{2\pi\left(\mu^{(2)}_{(1)}\right)^2\nu^3_{(1)}} - \frac{a_1a_2\,(d-2)\sin\left(\beta^{(1)}_{(1)}\right)\cos^2\left(\beta^{(1)}_{(1)}\right)}{2\pi\left(\mu^{(1)}_{(1)}\right)^2\nu^3_{(1)}} \\
&+ \frac{a_1\,(d-2)\sin\left(\beta^{(2)}_{(1)}\right)\cos\left(\beta^{(2)}_{(1)}\right)}{2\pi\left(\mu^{(2)}_{(1)}\right)^2\nu^2_{(1)}} + \frac{a_2^2\,(d-2)\cos\left(\alpha^{(2)}_{(1)}\right)}{2\pi\nu^2_{(1)}\nu^{(2)}_{(1)}} - \frac{a_2^2\,(d-2)}{\pi\nu^2_{(1)}\nu^{(2)}_{(1)}} \\
&+ \frac{a_2^2\left(d^2-4d+4\right)\sin\left(\alpha^{(2)}_{(1)}\right)}{2\pi\nu^2_{(1)}\left(\nu^{(2)}_{(1)}\right)^2} - \frac{a_2^2\left(d^2-3d+2\right)\sin\left(\alpha^{(2)}_{(1)}\right)\cos\left(\alpha^{(2)}_{(1)}\right)}{2\pi\nu^2_{(1)}\left(\nu^{(2)}_{(1)}\right)^2} \\
&+ \frac{a_2\,(d-2)\sin\left(\beta^{(1)}_{(1)}\right)\cos\left(\beta^{(1)}_{(1)}\right)}{2\pi\left(\mu^{(1)}_{(1)}\right)^2\nu^2_{(1)}} - \frac{\alpha^{(2)}_{(1)}\,(d-2)}{2\pi} + \frac{d}{2} - 1,
\end{aligned}
$$

$$
\begin{aligned}
T^{\mathfrak{s}}_{1,4} = &\frac{a_1\,(d-2)\sin\left(\alpha^{(2)}_{(1)}\right)}{2\pi} + a_1 - \frac{a_1\,(d-1)\sin\left(\beta^{(2)}_{(1)}\right)}{2\pi\nu_{(1)}} \\
&- \frac{a_1\sin\left(\beta^{(1)}_{(1)}\right)}{2\pi\nu_{(1)}} - \frac{a_2\alpha^{(2)}_{(1)}\,(d-2)}{2\pi} + \frac{a_2\,(d-2)}{2} + \frac{\beta^{(1)}_{(1)}}{2\pi} - \frac{1}{2},
\end{aligned}
$$

$$T^{\mathfrak{s}}_{2,1} = \frac{a_1^2 \cos\left(\alpha^{(2)}_{(1)}\right)}{4\pi\nu^2_{(1)}\nu^{(2)}_{(1)}} - \frac{a_1^2 \sin\left(\alpha^{(2)}_{(1)}\right)\cos\left(\alpha^{(2)}_{(1)}\right)}{4\pi\nu^2_{(1)}\left(\nu^{(2)}_{(1)}\right)^2} - \frac{a_1 a_2 (d-1)\sin\left(\alpha^{(2)}_{(1)}\right)}{4\pi\nu^2_{(1)}} - \frac{a_1 a_2}{2\pi\nu^2_{(1)}\nu^{(2)}_{(1)}}$$

$$- \frac{a_1 a_2 (d-2)\sin\left(\alpha^{(2)}_{(1)}\right)\cos\left(\alpha^{(2)}_{(1)}\right)}{4\pi\nu^2_{(1)}\left(\nu^{(2)}_{(1)}\right)^2} + \frac{a_1 a_2 (d-1)\sin\left(\alpha^{(2)}_{(1)}\right)\cos^2\left(\alpha^{(2)}_{(1)}\right)}{4\pi\nu^2_{(1)}\left(\nu^{(2)}_{(1)}\right)^2}$$

$$+ \frac{a_1 a_2 \sin\left(\alpha^{(2)}_{(1)}\right)}{4\pi\nu^2_{(1)}\left(\nu^{(2)}_{(1)}\right)^2} + \frac{a_1 a_2 (d-1)\sin\left(\beta^{(2)}_{(1)}\right)}{4\pi\nu^3_{(1)}} + \frac{a_1 a_2 \sin\left(\beta^{(1)}_{(1)}\right)}{4\pi\nu^3_{(1)}}$$

$$- \frac{a_1 a_2 (d-1)\sin\left(\beta^{(2)}_{(1)}\right)\cos^2\left(\beta^{(2)}_{(1)}\right)}{4\pi\left(\mu^{(2)}_{(1)}\right)^2\nu^3_{(1)}} - \frac{a_1 a_2 \sin\left(\beta^{(1)}_{(1)}\right)\cos^2\left(\beta^{(1)}_{(1)}\right)}{4\pi\left(\mu^{(1)}_{(1)}\right)^2\nu^3_{(1)}}$$

$$+ \frac{a_1 \sin\left(\beta^{(2)}_{(1)}\right)\cos\left(\beta^{(2)}_{(1)}\right)}{4\pi\left(\mu^{(2)}_{(1)}\right)^2\nu^2_{(1)}} + \frac{a_2^2 \cos\left(\alpha^{(2)}_{(1)}\right)}{4\pi\nu^2_{(1)}\nu^{(2)}_{(1)}} + \frac{a_2^2 (d-2)\sin\left(\alpha^{(2)}_{(1)}\right)}{4\pi\nu^2_{(1)}\left(\nu^{(2)}_{(1)}\right)^2}$$

$$- \frac{a_2^2 (d-1)\sin\left(\alpha^{(2)}_{(1)}\right)\cos\left(\alpha^{(2)}_{(1)}\right)}{4\pi\nu^2_{(1)}\left(\nu^{(2)}_{(1)}\right)^2} + \frac{a_2 \sin\left(\beta^{(1)}_{(1)}\right)\cos\left(\beta^{(1)}_{(1)}\right)}{4\pi\left(\mu^{(1)}_{(1)}\right)^2\nu^2_{(1)}} + \frac{\alpha^{(2)}_{(1)}}{4\pi} - \frac{1}{4},$$

$$T^{\mathfrak{s}}_{2,2} = -\frac{a_1^2}{2\pi\nu^2_{(1)}\nu^{(2)}_{(1)}} + \frac{a_1^2 \sin\left(\alpha^{(2)}_{(1)}\right)}{2\pi\nu^2_{(1)}\left(\nu^{(2)}_{(1)}\right)^2} + \frac{a_1 a_2 d \cos\left(\alpha^{(2)}_{(1)}\right)}{2\pi\nu^2_{(1)}\nu^{(2)}_{(1)}} - \frac{a_1 a_2 d \sin\left(\alpha^{(2)}_{(1)}\right)\cos\left(\alpha^{(2)}_{(1)}\right)}{2\pi\nu^2_{(1)}\left(\nu^{(2)}_{(1)}\right)^2}$$

$$- \frac{a_1 a_2 (d-2)}{2\pi\nu^2_{(1)}\nu^{(2)}_{(1)}} + \frac{a_1 a_2 (d-2)\sin\left(\alpha^{(2)}_{(1)}\right)}{2\pi\nu^2_{(1)}\left(\nu^{(2)}_{(1)}\right)^2} + \frac{a_2^2 d \sin\left(\beta^{(1)}_{(1)}\right)}{4\pi\nu^3_{(1)}}$$

$$- \frac{a_2^2 d \sin\left(\beta^{(1)}_{(1)}\right)\cos^2\left(\beta^{(1)}_{(1)}\right)}{4\pi\left(\mu^{(1)}_{(1)}\right)^2\nu^3_{(1)}} - \frac{a_2^2 (d^2-d)\sin\left(\alpha^{(2)}_{(1)}\right)}{4\pi\nu^2_{(1)}} - \frac{a_2^2}{2\pi\nu^2_{(1)}\nu^{(2)}_{(1)}}$$

$$- \frac{a_2^2 (d^2-2d)\sin\left(\alpha^{(2)}_{(1)}\right)\cos\left(\alpha^{(2)}_{(1)}\right)}{2\pi\nu^2_{(1)}\left(\nu^{(2)}_{(1)}\right)^2} + \frac{a_2^2 (d^2-d)\sin\left(\alpha^{(2)}_{(1)}\right)\cos^2\left(\alpha^{(2)}_{(1)}\right)}{4\pi\nu^2_{(1)}\left(\nu^{(2)}_{(1)}\right)^2}$$

$$+ \frac{a_2^2 (d^2-3d+2)\sin\left(\alpha^{(2)}_{(1)}\right)}{4\pi\nu^2_{(1)}\left(\nu^{(2)}_{(1)}\right)^2} + \frac{a_2^2 (d^2-d)\sin\left(\beta^{(2)}_{(1)}\right)}{4\pi\nu^3_{(1)}}$$

$$- \frac{a_2^2 (d^2-d)\sin\left(\beta^{(2)}_{(1)}\right)\cos^2\left(\beta^{(2)}_{(1)}\right)}{4\pi\left(\mu^{(2)}_{(1)}\right)^2\nu^3_{(1)}} + \frac{a_2 d \sin\left(\beta^{(2)}_{(1)}\right)\cos\left(\beta^{(2)}_{(1)}\right)}{2\pi\left(\mu^{(2)}_{(1)}\right)^2\nu^2_{(1)}} - \frac{\alpha^{(2)}_{(1)}(d-2)}{4\pi}$$

$$+ \frac{d}{4} + \frac{(d-1)\sin\left(\alpha^{(2)}_{(1)}\right)}{2\pi} - \frac{(d-1)\sin\left(\beta^{(2)}_{(1)}\right)}{2\pi\nu_{(1)}} - \frac{\sin\left(\beta^{(1)}_{(1)}\right)}{2\pi\nu_{(1)}} - \frac{\sin\left(\beta^{(2)}_{(1)}\right)}{2\pi\left(\mu^{(2)}_{(1)}\right)^2\nu_{(1)}},$$

$$\begin{aligned}
T^{\mathfrak{s}}_{2,3} = &-\frac{a_1 a_2 (d-2) \sin\left(\alpha^{(2)}_{(1)}\right) \cos\left(\alpha^{(2)}_{(1)}\right)}{2\pi \nu_{(1)}^2 \left(\nu^{(2)}_{(1)}\right)^2} + \frac{a_1 a_2 (d-2) \sin\left(\alpha^{(2)}_{(1)}\right)}{2\pi \nu_{(1)}^2 \left(\nu^{(2)}_{(1)}\right)^2} \\
&- \frac{a_2^2 \left(d^2 - 3d + 2\right) \sin\left(\alpha^{(2)}_{(1)}\right)}{4\pi \nu_{(1)}^2} + \frac{a_2^2 (d-2) \cos\left(\alpha^{(2)}_{(1)}\right)}{2\pi \nu_{(1)}^2 \nu^{(2)}_{(1)}} - \frac{a_2^2 (d-2)}{2\pi \nu_{(1)}^2 \nu^{(2)}_{(1)}} \\
&+ \frac{a_2^2 \left(d^2 - 5d + 6\right) \sin\left(\alpha^{(2)}_{(1)}\right)}{4\pi \nu_{(1)}^2 \left(\nu^{(2)}_{(1)}\right)^2} - \frac{a_2^2 \left(d^2 - 4d + 4\right) \sin\left(\alpha^{(2)}_{(1)}\right) \cos\left(\alpha^{(2)}_{(1)}\right)}{2\pi \nu_{(1)}^2 \left(\nu^{(2)}_{(1)}\right)^2} \\
&+ \frac{a_2^2 \left(d^2 - 3d + 2\right) \sin\left(\alpha^{(2)}_{(1)}\right) \cos^2\left(\alpha^{(2)}_{(1)}\right)}{4\pi \nu_{(1)}^2 \left(\nu^{(2)}_{(1)}\right)^2} \\
&+ \frac{a_2^2 (d-2) \sin\left(\beta^{(1)}_{(1)}\right)}{4\pi \nu_{(1)}^3} + \frac{a_2^2 \left(d^2 - 3d + 2\right) \sin\left(\beta^{(2)}_{(1)}\right)}{4\pi \nu_{(1)}^3} \\
&- \frac{a_2^2 \left(d^2 - 3d + 2\right) \sin\left(\beta^{(2)}_{(1)}\right) \cos^2\left(\beta^{(2)}_{(1)}\right)}{4\pi \left(\mu^{(2)}_{(1)}\right)^2 \nu_{(1)}^3} - \frac{a_2^2 (d-2) \sin\left(\beta^{(1)}_{(1)}\right) \cos^2\left(\beta^{(1)}_{(1)}\right)}{4\pi \left(\mu^{(1)}_{(1)}\right)^2 \nu_{(1)}^3} \\
&+ \frac{a_2 (d-2) \sin\left(\beta^{(2)}_{(1)}\right) \cos\left(\beta^{(2)}_{(1)}\right)}{2\pi \left(\mu^{(2)}_{(1)}\right)^2 \nu_{(1)}^2} + \frac{\alpha^{(2)}_{(1)} (d-2)}{4\pi} - \frac{d}{4} + \frac{1}{2},
\end{aligned}$$

$$\begin{aligned}
T^{\mathfrak{s}}_{2,4} = &-\frac{a_2 \alpha^{(2)}_{(1)} (d-2)}{4\pi} + \frac{a_2 d}{4} + \frac{a_2 (d-2) \sin\left(\alpha^{(2)}_{(1)}\right)}{4\pi} \\
&- \frac{a_2 (d-1) \sin\left(\beta^{(2)}_{(1)}\right)}{4\pi \nu_{(1)}} - \frac{a_2 \sin\left(\beta^{(1)}_{(1)}\right)}{4\pi \nu_{(1)}} + \frac{\beta^{(2)}_{(1)}}{4\pi} - \frac{1}{4},
\end{aligned}$$

$$\begin{aligned}
T^{\mathfrak{s}}_{3,1} = &-\frac{a_1^2 \cos\left(\alpha^{(2)}_{(1)}\right)}{4\pi \nu_{(1)}^2 \nu^{(2)}_{(1)}} - \frac{a_1^2 \sin\left(\alpha^{(2)}_{(1)}\right) \cos\left(\alpha^{(2)}_{(1)}\right)}{4\pi \nu_{(1)}^2 \left(\nu^{(2)}_{(1)}\right)^2} - \frac{a_1 a_2 (d-1) \sin\left(\alpha^{(2)}_{(1)}\right)}{4\pi \nu_{(1)}^2} + \frac{a_1 a_2 \cos\left(\alpha^{(2)}_{(1)}\right)}{2\pi \nu_{(1)}^2 \nu^{(2)}_{(1)}} \\
&- \frac{a_1 a_2 (d-2) \sin\left(\alpha^{(2)}_{(1)}\right) \cos\left(\alpha^{(2)}_{(1)}\right)}{4\pi \nu_{(1)}^2 \left(\nu^{(2)}_{(1)}\right)^2} + \frac{a_1 a_2 (d-1) \sin\left(\alpha^{(2)}_{(1)}\right) \cos^2\left(\alpha^{(2)}_{(1)}\right)}{4\pi \nu_{(1)}^2 \left(\nu^{(2)}_{(1)}\right)^2} \\
&+ \frac{a_1 a_2 \sin\left(\alpha^{(2)}_{(1)}\right)}{4\pi \nu_{(1)}^2 \left(\nu^{(2)}_{(1)}\right)^2} + \frac{a_1 a_2 (d-1) \sin\left(\beta^{(2)}_{(1)}\right)}{4\pi \nu_{(1)}^3} + \frac{a_1 a_2 \sin\left(\beta^{(1)}_{(1)}\right)}{4\pi \nu_{(1)}^3} \\
&- \frac{a_1 a_2 (d-1) \sin\left(\beta^{(2)}_{(1)}\right) \cos^2\left(\beta^{(2)}_{(1)}\right)}{4\pi \left(\mu^{(2)}_{(1)}\right)^2 \nu_{(1)}^3} - \frac{a_1 a_2 \sin\left(\beta^{(1)}_{(1)}\right) \cos^2\left(\beta^{(1)}_{(1)}\right)}{4\pi \left(\mu^{(1)}_{(1)}\right)^2 \nu_{(1)}^3} \\
&+ \frac{a_1 \sin\left(\beta^{(2)}_{(1)}\right) \cos\left(\beta^{(2)}_{(1)}\right)}{4\pi \left(\mu^{(2)}_{(1)}\right)^2 \nu_{(1)}^2} + \frac{a_2^2 \cos\left(\alpha^{(2)}_{(1)}\right)}{4\pi \nu_{(1)}^2 \nu^{(2)}_{(1)}} - \frac{a_2^2}{2\pi \nu_{(1)}^2 \nu^{(2)}_{(1)}} + \frac{a_2^2 (d-2) \sin\left(\alpha^{(2)}_{(1)}\right)}{4\pi \nu_{(1)}^2 \left(\nu^{(2)}_{(1)}\right)^2} \\
&- \frac{a_2^2 (d-1) \sin\left(\alpha^{(2)}_{(1)}\right) \cos\left(\alpha^{(2)}_{(1)}\right)}{4\pi \nu_{(1)}^2 \left(\nu^{(2)}_{(1)}\right)^2} + \frac{a_2 \sin\left(\beta^{(1)}_{(1)}\right) \cos\left(\beta^{(1)}_{(1)}\right)}{4\pi \left(\mu^{(1)}_{(1)}\right)^2 \nu_{(1)}^2} - \frac{\alpha^{(2)}_{(1)}}{4\pi} + \frac{1}{4},
\end{aligned}$$

$$
\begin{aligned}
T^{\mathfrak{s}}_{3,2} = &-\frac{a_1 a_2 d \sin\left(\alpha^{(2)}_{(1)}\right)\cos\left(\alpha^{(2)}_{(1)}\right)}{2\pi\nu^2_{(1)}\left(\nu^{(2)}_{(1)}\right)^2} + \frac{a_1 a_2 d \sin\left(\alpha^{(2)}_{(1)}\right)}{2\pi\nu^2_{(1)}\left(\nu^{(2)}_{(1)}\right)^2} + \frac{a_2^2 d \cos\left(\alpha^{(2)}_{(1)}\right)}{2\pi\nu^2_{(1)}\nu^{(2)}_{(1)}} - \frac{a_2^2 d}{2\pi\nu^2_{(1)}\nu^{(2)}_{(1)}} \\
&+ \frac{a_2^2 d \sin\left(\beta^{(1)}_{(1)}\right)}{4\pi\nu^3_{(1)}} - \frac{a_2^2 d \sin\left(\beta^{(1)}_{(1)}\right)\cos^2\left(\beta^{(1)}_{(1)}\right)}{4\pi\left(\mu^{(1)}_{(1)}\right)^2\nu^3_{(1)}} - \frac{a_2^2\left(d^2-d\right)\sin\left(\alpha^{(2)}_{(1)}\right)}{4\pi\nu^2_{(1)}} \\
&+ \frac{a_2^2\left(d^2-3d\right)\sin\left(\alpha^{(2)}_{(1)}\right)}{4\pi\nu^2_{(1)}\left(\nu^{(2)}_{(1)}\right)^2} - \frac{a_2^2\left(d^2-2d\right)\sin\left(\alpha^{(2)}_{(1)}\right)\cos\left(\alpha^{(2)}_{(1)}\right)}{2\pi\nu^2_{(1)}\left(\nu^{(2)}_{(1)}\right)^2} \\
&+ \frac{a_2^2\left(d^2-d\right)\sin\left(\alpha^{(2)}_{(1)}\right)\cos^2\left(\alpha^{(2)}_{(1)}\right)}{4\pi\nu^2_{(1)}\left(\nu^{(2)}_{(1)}\right)^2} + \frac{a_2^2\left(d^2-d\right)\sin\left(\beta^{(2)}_{(1)}\right)}{4\pi\nu^3_{(1)}} \\
&- \frac{a_2^2\left(d^2-d\right)\sin\left(\beta^{(2)}_{(1)}\right)\cos^2\left(\beta^{(2)}_{(1)}\right)}{4\pi\left(\mu^{(2)}_{(1)}\right)^2\nu^3_{(1)}} + \frac{a_2 d \sin\left(\beta^{(2)}_{(1)}\right)\cos\left(\beta^{(2)}_{(1)}\right)}{2\pi\left(\mu^{(2)}_{(1)}\right)^2\nu^2_{(1)}} + \frac{\alpha^{(2)}_{(1)} d}{4\pi} - \frac{d}{4},
\end{aligned}
$$

$$
\begin{aligned}
T^{\mathfrak{s}}_{3,3} = &\frac{a_1^2}{2\pi\nu^2_{(1)}\nu^{(2)}_{(1)}} + \frac{a_1^2 \sin\left(\alpha^{(2)}_{(1)}\right)}{2\pi\nu^2_{(1)}\left(\nu^{(2)}_{(1)}\right)^2} + \frac{a_1 a_2 (d-4)}{2\pi\nu^2_{(1)}\nu^{(2)}_{(1)}} - \frac{a_1 a_2 (d-2)\cos\left(\alpha^{(2)}_{(1)}\right)}{2\pi\nu^2_{(1)}\nu^{(2)}_{(1)}} \\
&+ \frac{a_1 a_2 (d-4)\sin\left(\alpha^{(2)}_{(1)}\right)}{2\pi\nu^2_{(1)}\left(\nu^{(2)}_{(1)}\right)^2} - \frac{a_1 a_2 (d-2)\sin\left(\alpha^{(2)}_{(1)}\right)\cos\left(\alpha^{(2)}_{(1)}\right)}{2\pi\nu^2_{(1)}\left(\nu^{(2)}_{(1)}\right)^2} \\
&- \frac{a_2^2\left(d^2-3d+2\right)\sin\left(\alpha^{(2)}_{(1)}\right)}{4\pi\nu^2_{(1)}} - \frac{a_2^2\left(d-\frac{5}{2}\right)}{\pi\nu^2_{(1)}\nu^{(2)}_{(1)}} + \frac{a_2^2 (d-2)\cos\left(\alpha^{(2)}_{(1)}\right)}{\pi\nu^2_{(1)}\nu^{(2)}_{(1)}} \\
&+ \frac{a_2^2\left(d^2-5d+8\right)\sin\left(\alpha^{(2)}_{(1)}\right)}{4\pi\nu^2_{(1)}\left(\nu^{(2)}_{(1)}\right)^2} - \frac{a_2^2\left(d^2-4d+4\right)\sin\left(\alpha^{(2)}_{(1)}\right)\cos\left(\alpha^{(2)}_{(1)}\right)}{2\pi\nu^2_{(1)}\left(\nu^{(2)}_{(1)}\right)^2} \\
&+ \frac{a_2^2\left(d^2-3d+2\right)\sin\left(\alpha^{(2)}_{(1)}\right)\cos^2\left(\alpha^{(2)}_{(1)}\right)}{4\pi\nu^2_{(1)}\left(\nu^{(2)}_{(1)}\right)^2} + \frac{a_2^2 (d-2)\sin\left(\beta^{(1)}_{(1)}\right)}{4\pi\nu^3_{(1)}} \\
&+ \frac{a_2^2\left(d^2-3d+2\right)\sin\left(\beta^{(2)}_{(1)}\right)}{4\pi\nu^3_{(1)}} - \frac{a_2^2\left(d^2-3d+2\right)\sin\left(\beta^{(2)}_{(1)}\right)\cos^2\left(\beta^{(2)}_{(1)}\right)}{4\pi\left(\mu^{(2)}_{(1)}\right)^2\nu^3_{(1)}} \\
&- \frac{a_2^2 (d-2)\sin\left(\beta^{(1)}_{(1)}\right)\cos^2\left(\beta^{(1)}_{(1)}\right)}{4\pi\left(\mu^{(1)}_{(1)}\right)^2\nu^3_{(1)}} + \frac{a_2 (d-2)\sin\left(\beta^{(2)}_{(1)}\right)\cos\left(\beta^{(2)}_{(1)}\right)}{2\pi\left(\mu^{(2)}_{(1)}\right)^2\nu^2_{(1)}} - \frac{\alpha^{(2)}_{(1)}(d-4)}{4\pi} \\
&+ \frac{d}{4} + \frac{(d-1)\sin\left(\alpha^{(2)}_{(1)}\right)}{2\pi} - \frac{1}{2} - \frac{(d-1)\sin\left(\beta^{(2)}_{(1)}\right)}{2\pi\nu_{(1)}} - \frac{\sin\left(\beta^{(1)}_{(1)}\right)}{2\pi\nu_{(1)}} - \frac{\sin\left(\beta^{(2)}_{(1)}\right)}{2\pi\left(\mu^{(2)}_{(1)}\right)^2\nu_{(1)}},
\end{aligned}
$$

$$
\begin{aligned}
T^{\mathfrak{s}}_{3,4} = &-\frac{a_1\alpha^{(2)}_{(1)}}{2\pi} + \frac{a_1}{2} - \frac{a_2\alpha^{(2)}_{(1)}(d-4)}{4\pi} + \frac{a_2 (d-2)\sin\left(\alpha^{(2)}_{(1)}\right)}{4\pi} \\
&+ \frac{a_2 (d-2)}{4} - \frac{a_2 (d-1)\sin\left(\beta^{(2)}_{(1)}\right)}{4\pi\nu_{(1)}} - \frac{a_2 \sin\left(\beta^{(1)}_{(1)}\right)}{4\pi\nu_{(1)}} + \frac{\beta^{(2)}_{(1)}}{4\pi} - \frac{1}{4},
\end{aligned}
$$

$$T^{\mathfrak{s}}_{4,1} = \frac{a_1\left(d-2\right)\sin\left(\alpha^{(2)}_{(1)}\right)}{2\pi} + a_1 - \frac{a_1\left(d-1\right)\sin\left(\beta^{(2)}_{(1)}\right)}{2\pi\nu_{(1)}}$$
$$- \frac{a_1\sin\left(\beta^{(1)}_{(1)}\right)}{2\pi\nu_{(1)}} - \frac{a_2\alpha^{(2)}_{(1)}\left(d-2\right)}{2\pi} + \frac{a_2\left(d-2\right)}{2} + \frac{\beta^{(1)}_{(1)}}{2\pi} - \frac{1}{2},$$

$$T^{\mathfrak{s}}_{4,2} = -\frac{a_2\alpha^{(2)}_{(1)}\left(d^2-2d\right)}{2\pi} + \frac{a_2 d^2}{2} - \frac{a_2 d\sin\left(\beta^{(1)}_{(1)}\right)}{2\pi\nu_{(1)}}$$
$$+ \frac{a_2\left(d^2-2d\right)\sin\left(\alpha^{(2)}_{(1)}\right)}{2\pi} - \frac{a_2\left(d^2-d\right)\sin\left(\beta^{(2)}_{(1)}\right)}{2\pi\nu_{(1)}} + \frac{\beta^{(2)}_{(1)} d}{2\pi} - \frac{d}{2},$$

$$T^{\mathfrak{s}}_{4,3} = -\frac{a_1\alpha^{(2)}_{(1)}\left(d-2\right)}{\pi} + a_1\left(d-2\right) - \frac{a_2\alpha^{(2)}_{(1)}\left(d^2-6d+8\right)}{2\pi}$$
$$+ \frac{a_2\left(d^2-4d+4\right)\sin\left(\alpha^{(2)}_{(1)}\right)}{2\pi} + \frac{a_2\left(d^2-4d+4\right)}{2} - \frac{a_2\left(d-2\right)\sin\left(\beta^{(1)}_{(1)}\right)}{2\pi\nu_{(1)}}$$
$$- \frac{a_2\left(d^2-3d+2\right)\sin\left(\beta^{(2)}_{(1)}\right)}{2\pi\nu_{(1)}} + \frac{\beta^{(2)}_{(1)}\left(d-2\right)}{2\pi} - \frac{d}{2} + 1,$$

$$T^{\mathfrak{s}}_{4,4} = \frac{\alpha^{(2)}_{(1)}\nu^2_{(1)}\cos\left(\alpha^{(2)}_{(1)}\right)}{2\pi} - \frac{\nu^2_{(1)}\sin\left(\alpha^{(2)}_{(1)}\right)}{2\pi} - \frac{\nu^2_{(1)}\cos\left(\alpha^{(2)}_{(1)}\right)}{2} + \frac{\nu^2_{(1)}}{2}.$$

$\mathfrak{t}$-rep. Let the associated 3×3 transition matrix be denoted by $T^{\mathfrak{t}}$. Then,

$$T^{\mathfrak{t}}_{1,1} = -\frac{a_1^2\left(d-1\right)\sin\left(\alpha^{(2)}_{(1)}\right)}{2\pi\nu^2_{(1)}} + \frac{a_1^2\left(d-1\right)}{2\pi\nu^2_{(1)}\nu^{(2)}_{(1)}} + \frac{a_1^2\left(d-1\right)\sin\left(\alpha^{(2)}_{(1)}\right)\cos^2\left(\alpha^{(2)}_{(1)}\right)}{2\pi\nu^2_{(1)}\left(\nu^{(2)}_{(1)}\right)^2}$$
$$+ \frac{a_1^2\left(d-1\right)\sin\left(\beta^{(2)}_{(1)}\right)}{2\pi\nu^3_{(1)}} + \frac{a_1^2\sin\left(\beta^{(1)}_{(1)}\right)}{2\pi\nu^3_{(1)}} - \frac{a_1^2\left(d-1\right)\sin\left(\beta^{(2)}_{(1)}\right)\cos^2\left(\beta^{(2)}_{(1)}\right)}{2\pi\left(\mu^{(2)}_{(1)}\right)^2\nu^3_{(1)}}$$
$$- \frac{a_1^2\sin\left(\beta^{(1)}_{(1)}\right)\cos^2\left(\beta^{(1)}_{(1)}\right)}{2\pi\left(\mu^{(1)}_{(1)}\right)^2\nu^3_{(1)}} - \frac{a_1 a_2\left(d-1\right)\cos\left(\alpha^{(2)}_{(1)}\right)}{\pi\nu^2_{(1)}\nu^{(2)}_{(1)}}$$
$$- \frac{a_1 a_2\left(d-1\right)\sin\left(\alpha^{(2)}_{(1)}\right)\cos\left(\alpha^{(2)}_{(1)}\right)}{\pi\nu^2_{(1)}\left(\nu^{(2)}_{(1)}\right)^2} + \frac{a_1\sin\left(\beta^{(1)}_{(1)}\right)\cos\left(\beta^{(1)}_{(1)}\right)}{\pi\left(\mu^{(1)}_{(1)}\right)^2\nu^2_{(1)}}$$
$$+ \frac{a_2^2\left(d-1\right)}{2\pi\nu^2_{(1)}\nu^{(2)}_{(1)}} + \frac{a_2^2\left(d-1\right)\sin\left(\alpha^{(2)}_{(1)}\right)}{2\pi\nu^2_{(1)}\left(\nu^{(2)}_{(1)}\right)^2} + \frac{\left(d-1\right)\sin\left(\alpha^{(2)}_{(1)}\right)}{2\pi}$$
$$+ \frac{1}{2} - \frac{\left(d-1\right)\sin\left(\beta^{(2)}_{(1)}\right)}{2\pi\nu_{(1)}} - \frac{\sin\left(\beta^{(1)}_{(1)}\right)}{2\pi\nu_{(1)}} - \frac{\sin\left(\beta^{(1)}_{(1)}\right)}{2\pi\left(\mu^{(1)}_{(1)}\right)^2\nu_{(1)}},$$

$$T^{\mathrm{t}}_{1,2} = -\frac{a_1^2\left(d-1\right)\cos\left(\alpha^{(2)}_{(1)}\right)}{2\pi\nu_{(1)}^2\nu^{(2)}_{(1)}} - \frac{a_1^2\left(d-1\right)\sin\left(\alpha^{(2)}_{(1)}\right)\cos\left(\alpha^{(2)}_{(1)}\right)}{2\pi\nu_{(1)}^2\left(\nu^{(2)}_{(1)}\right)^2}$$

$$- \frac{a_1a_2\left(d^2-2d+1\right)\sin\left(\alpha^{(2)}_{(1)}\right)}{2\pi\nu_{(1)}^2} + \frac{a_1a_2\left(d^2-d\right)}{2\pi\nu_{(1)}^2\nu^{(2)}_{(1)}} - \frac{a_1a_2\left(d^2-3d+2\right)\cos\left(\alpha^{(2)}_{(1)}\right)}{2\pi\nu_{(1)}^2\nu^{(2)}_{(1)}}$$

$$+ \frac{a_1a_2\left(d-1\right)\sin\left(\alpha^{(2)}_{(1)}\right)}{2\pi\nu_{(1)}^2\left(\nu^{(2)}_{(1)}\right)^2} - \frac{a_1a_2\left(d^2-3d+2\right)\sin\left(\alpha^{(2)}_{(1)}\right)\cos\left(\alpha^{(2)}_{(1)}\right)}{2\pi\nu_{(1)}^2\left(\nu^{(2)}_{(1)}\right)^2}$$

$$+ \frac{a_1a_2\left(d^2-2d+1\right)\sin\left(\alpha^{(2)}_{(1)}\right)\cos^2\left(\alpha^{(2)}_{(1)}\right)}{2\pi\nu_{(1)}^2\left(\nu^{(2)}_{(1)}\right)^2} + \frac{a_1a_2\left(d-1\right)\sin\left(\beta^{(1)}_{(1)}\right)}{2\pi\nu_{(1)}^3}$$

$$+ \frac{a_1a_2\left(d^2-2d+1\right)\sin\left(\beta^{(2)}_{(1)}\right)}{2\pi\nu_{(1)}^3} - \frac{a_1a_2\left(d^2-2d+1\right)\sin\left(\beta^{(2)}_{(1)}\right)\cos^2\left(\beta^{(2)}_{(1)}\right)}{2\pi\left(\mu^{(2)}_{(1)}\right)^2\nu_{(1)}^3}$$

$$- \frac{a_1a_2\left(d-1\right)\sin\left(\beta^{(1)}_{(1)}\right)\cos^2\left(\beta^{(1)}_{(1)}\right)}{2\pi\left(\mu^{(1)}_{(1)}\right)^2\nu_{(1)}^3} + \frac{a_1\left(d-1\right)\sin\left(\beta^{(2)}_{(1)}\right)\cos\left(\beta^{(2)}_{(1)}\right)}{2\pi\left(\mu^{(2)}_{(1)}\right)^2\nu_{(1)}^2}$$

$$+ \frac{a_2^2\left(d^2-3d+2\right)}{2\pi\nu_{(1)}^2\nu^{(2)}_{(1)}} - \frac{a_2^2\left(d^2-2d+1\right)\cos\left(\alpha^{(2)}_{(1)}\right)}{2\pi\nu_{(1)}^2\nu^{(2)}_{(1)}}$$

$$+ \frac{a_2^2\left(d^2-3d+2\right)\sin\left(\alpha^{(2)}_{(1)}\right)}{2\pi\nu_{(1)}^2\left(\nu^{(2)}_{(1)}\right)^2} - \frac{a_2^2\left(d^2-2d+1\right)\sin\left(\alpha^{(2)}_{(1)}\right)\cos\left(\alpha^{(2)}_{(1)}\right)}{2\pi\nu_{(1)}^2\left(\nu^{(2)}_{(1)}\right)^2}$$

$$+ \frac{a_2\left(d-1\right)\sin\left(\beta^{(1)}_{(1)}\right)\cos\left(\beta^{(1)}_{(1)}\right)}{2\pi\left(\mu^{(1)}_{(1)}\right)^2\nu_{(1)}^2} - \frac{\alpha^{(2)}_{(1)}\left(d-1\right)}{2\pi} + \frac{d}{2} - \frac{1}{2},$$

$$T^{\mathrm{t}}_{1,3} = \frac{a_1\left(d-1\right)\sin\left(\alpha^{(2)}_{(1)}\right)}{\pi} + a_1 - \frac{a_1\left(d-1\right)\sin\left(\beta^{(2)}_{(1)}\right)}{2\pi\nu_{(1)}}$$

$$- \frac{a_1\sin\left(\beta^{(1)}_{(1)}\right)}{2\pi\nu_{(1)}} - \frac{a_2\alpha^{(2)}_{(1)}\left(d-1\right)}{\pi} + a_2\left(d-1\right) + \frac{\beta^{(1)}_{(1)}}{2\pi} - \frac{1}{2},$$

$$
\begin{aligned}
T^{\mathrm{t}}_{2,1} = &-\frac{a_1^2\cos\left(\alpha^{(2)}_{(1)}\right)}{2\pi\nu^2_{(1)}\nu^{(2)}_{(1)}} - \frac{a_1^2\sin\left(\alpha^{(2)}_{(1)}\right)\cos\left(\alpha^{(2)}_{(1)}\right)}{2\pi\nu^2_{(1)}\left(\nu^{(2)}_{(1)}\right)^2} + \frac{a_1a_2d}{2\pi\nu^2_{(1)}\nu^{(2)}_{(1)}} - \frac{a_1a_2\left(d-1\right)\sin\left(\alpha^{(2)}_{(1)}\right)}{2\pi\nu^2_{(1)}} \\
&- \frac{a_1a_2\left(d-2\right)\cos\left(\alpha^{(2)}_{(1)}\right)}{2\pi\nu^2_{(1)}\nu^{(2)}_{(1)}} - \frac{a_1a_2\left(d-2\right)\sin\left(\alpha^{(2)}_{(1)}\right)\cos\left(\alpha^{(2)}_{(1)}\right)}{2\pi\nu^2_{(1)}\left(\nu^{(2)}_{(1)}\right)^2} \\
&+ \frac{a_1a_2\left(d-1\right)\sin\left(\alpha^{(2)}_{(1)}\right)\cos^2\left(\alpha^{(2)}_{(1)}\right)}{2\pi\nu^2_{(1)}\left(\nu^{(2)}_{(1)}\right)^2} + \frac{a_1a_2\sin\left(\alpha^{(2)}_{(1)}\right)}{2\pi\nu^2_{(1)}\left(\nu^{(2)}_{(1)}\right)^2} + \frac{a_1a_2\left(d-1\right)\sin\left(\beta^{(2)}_{(1)}\right)}{2\pi\nu^3_{(1)}} \\
&+ \frac{a_1a_2\sin\left(\beta^{(1)}_{(1)}\right)}{2\pi\nu^3_{(1)}} - \frac{a_1a_2\left(d-1\right)\sin\left(\beta^{(2)}_{(1)}\right)\cos^2\left(\beta^{(2)}_{(1)}\right)}{2\pi\left(\mu^{(2)}_{(1)}\right)^2\nu^3_{(1)}} - \frac{a_1a_2\sin\left(\beta^{(1)}_{(1)}\right)\cos^2\left(\beta^{(1)}_{(1)}\right)}{2\pi\left(\mu^{(1)}_{(1)}\right)^2\nu^3_{(1)}} \\
&+ \frac{a_1\sin\left(\beta^{(2)}_{(1)}\right)\cos\left(\beta^{(2)}_{(1)}\right)}{2\pi\left(\mu^{(2)}_{(1)}\right)^2\nu^2_{(1)}} + \frac{a_2^2\left(d-2\right)}{2\pi\nu^2_{(1)}\nu^{(2)}_{(1)}} - \frac{a_2^2\left(d-1\right)\cos\left(\alpha^{(2)}_{(1)}\right)}{2\pi\nu^2_{(1)}\nu^{(2)}_{(1)}} + \frac{a_2^2\left(d-2\right)\sin\left(\alpha^{(2)}_{(1)}\right)}{2\pi\nu^2_{(1)}\left(\nu^{(2)}_{(1)}\right)^2} \\
&- \frac{a_2^2\left(d-1\right)\sin\left(\alpha^{(2)}_{(1)}\right)\cos\left(\alpha^{(2)}_{(1)}\right)}{2\pi\nu^2_{(1)}\left(\nu^{(2)}_{(1)}\right)^2} + \frac{a_2\sin\left(\beta^{(1)}_{(1)}\right)\cos\left(\beta^{(1)}_{(1)}\right)}{2\pi\left(\mu^{(1)}_{(1)}\right)^2\nu^2_{(1)}} - \frac{\alpha^{(2)}_{(1)}}{2\pi} + \frac{1}{2},
\end{aligned}
$$

$$
\begin{aligned}
T^{\mathrm{t}}_{2,2} = &\frac{a_1^2}{2\pi\nu^2_{(1)}\nu^{(2)}_{(1)}} + \frac{a_1^2\sin\left(\alpha^{(2)}_{(1)}\right)}{2\pi\nu^2_{(1)}\left(\nu^{(2)}_{(1)}\right)^2} + \frac{a_1a_2\left(d-2\right)}{\pi\nu^2_{(1)}\nu^{(2)}_{(1)}} - \frac{a_1a_2\left(d-1\right)\cos\left(\alpha^{(2)}_{(1)}\right)}{\pi\nu^2_{(1)}\nu^{(2)}_{(1)}} \\
&+ \frac{a_1a_2\left(d-2\right)\sin\left(\alpha^{(2)}_{(1)}\right)}{\pi\nu^2_{(1)}\left(\nu^{(2)}_{(1)}\right)^2} - \frac{a_1a_2\left(d-1\right)\sin\left(\alpha^{(2)}_{(1)}\right)\cos\left(\alpha^{(2)}_{(1)}\right)}{\pi\nu^2_{(1)}\left(\nu^{(2)}_{(1)}\right)^2} \\
&- \frac{a_2^2\left(d^2-2d+1\right)\sin\left(\alpha^{(2)}_{(1)}\right)}{2\pi\nu^2_{(1)}} - \frac{a_2^2\left(d^2-3d+2\right)\cos\left(\alpha^{(2)}_{(1)}\right)}{\pi\nu^2_{(1)}\nu^{(2)}_{(1)}} + \frac{a_2^2\left(d^2-3d+\frac{5}{2}\right)}{\pi\nu^2_{(1)}\nu^{(2)}_{(1)}} \\
&+ \frac{a_2^2\left(d^2-4d+4\right)\sin\left(\alpha^{(2)}_{(1)}\right)}{2\pi\nu^2_{(1)}\left(\nu^{(2)}_{(1)}\right)^2} - \frac{a_2^2\left(d^2-3d+2\right)\sin\left(\alpha^{(2)}_{(1)}\right)\cos\left(\alpha^{(2)}_{(1)}\right)}{\pi\nu^2_{(1)}\left(\nu^{(2)}_{(1)}\right)^2} \\
&+ \frac{a_2^2\left(d^2-2d+1\right)\sin\left(\alpha^{(2)}_{(1)}\right)\cos^2\left(\alpha^{(2)}_{(1)}\right)}{2\pi\nu^2_{(1)}\left(\nu^{(2)}_{(1)}\right)^2} + \frac{a_2^2\left(d-1\right)\sin\left(\beta^{(1)}_{(1)}\right)}{2\pi\nu^3_{(1)}} \\
&+ \frac{a_2^2\left(d^2-2d+1\right)\sin\left(\beta^{(2)}_{(1)}\right)}{2\pi\nu^3_{(1)}} - \frac{a_2^2\left(d^2-2d+1\right)\sin\left(\beta^{(2)}_{(1)}\right)\cos^2\left(\beta^{(2)}_{(1)}\right)}{2\pi\left(\mu^{(2)}_{(1)}\right)^2\nu^3_{(1)}} \\
&- \frac{a_2^2\left(d-1\right)\sin\left(\beta^{(1)}_{(1)}\right)\cos^2\left(\beta^{(1)}_{(1)}\right)}{2\pi\left(\mu^{(1)}_{(1)}\right)^2\nu^3_{(1)}} + \frac{a_2\left(d-1\right)\sin\left(\beta^{(2)}_{(1)}\right)\cos\left(\beta^{(2)}_{(1)}\right)}{\pi\left(\mu^{(2)}_{(1)}\right)^2\nu^2_{(1)}} - \frac{\alpha^{(2)}_{(1)}\left(d-2\right)}{2\pi} \\
&+ \frac{d}{2} + \frac{\left(d-1\right)\sin\left(\alpha^{(2)}_{(1)}\right)}{2\pi} - \frac{1}{2} - \frac{\left(d-1\right)\sin\left(\beta^{(2)}_{(1)}\right)}{2\pi\nu_{(1)}} - \frac{\sin\left(\beta^{(1)}_{(1)}\right)}{2\pi\nu_{(1)}} - \frac{\sin\left(\beta^{(2)}_{(1)}\right)}{2\pi\left(\mu^{(2)}_{(1)}\right)^2\nu_{(1)}},
\end{aligned}
$$

$$T^{\mathrm{t}}_{2,3} = -\frac{a_1 \alpha^{(2)}_{(1)}}{\pi} + a_1 - \frac{a_2 \alpha^{(2)}_{(1)} (d-2)}{\pi} + \frac{a_2 (d-1) \sin\left(\alpha^{(2)}_{(1)}\right)}{\pi}$$
$$+ a_2 (d-1) - \frac{a_2 (d-1) \sin\left(\beta^{(2)}_{(1)}\right)}{2\pi \nu_{(1)}} - \frac{a_2 \sin\left(\beta^{(1)}_{(1)}\right)}{2\pi \nu_{(1)}} + \frac{\beta^{(2)}_{(1)}}{2\pi} - \frac{1}{2},$$

$$T^{\mathrm{t}}_{3,1} = \frac{a_1 (d-1) \sin\left(\alpha^{(2)}_{(1)}\right)}{\pi} + a_1 - \frac{a_1 (d-1) \sin\left(\beta^{(2)}_{(1)}\right)}{2\pi \nu_{(1)}}$$
$$- \frac{a_1 \sin\left(\beta^{(1)}_{(1)}\right)}{2\pi \nu_{(1)}} - \frac{a_2 \alpha^{(2)}_{(1)} (d-1)}{\pi} + a_2 (d-1) + \frac{\beta^{(1)}_{(1)}}{2\pi} - \frac{1}{2},$$

$$T^{\mathrm{t}}_{3,2} = -\frac{a_1 \alpha^{(2)}_{(1)} (d-1)}{\pi} + a_1 (d-1) - \frac{a_2 \alpha^{(2)}_{(1)} \left(d^2 - 3d + 2\right)}{\pi}$$
$$+ \frac{a_2 \left(d^2 - 2d + 1\right) \sin\left(\alpha^{(2)}_{(1)}\right)}{\pi} + a_2 \left(d^2 - 2d + 1\right) - \frac{a_2 (d-1) \sin\left(\beta^{(1)}_{(1)}\right)}{2\pi \nu_{(1)}}$$
$$- \frac{a_2 \left(d^2 - 2d + 1\right) \sin\left(\beta^{(2)}_{(1)}\right)}{2\pi \nu_{(1)}} + \frac{\beta^{(2)}_{(1)} (d-1)}{2\pi} - \frac{d}{2} + \frac{1}{2},$$

$$T^{\mathrm{t}}_{3,3} = -\frac{\alpha^{(2)}_{(1)} \nu^2_{(1)} (d-1) \cos\left(\alpha^{(2)}_{(1)}\right)}{2\pi} + \frac{\nu^2_{(1)} (d-1) \sin\left(\alpha^{(2)}_{(1)}\right)}{2\pi} + \frac{\nu^2_{(1)} (d-1) \cos\left(\alpha^{(2)}_{(1)}\right)}{2} + \frac{\nu^2_{(1)}}{2}.$$

F Power series and Hessian spectrum by representation

We present the (fractional) power series of the minima described in Theorem 1 to $O(d^{-5/2})$-order, along with the respective eigenvalues arranged by their representation.

F.1 Theorem 1, case 2a

$$\begin{aligned}
a_1 &= 1 + \frac{8}{\pi d^2} + O\left(d^{\frac{-5}{2}}\right), \\
a_2 &= -\frac{4}{\pi d^2} + O\left(d^{\frac{-5}{2}}\right), \\
a_3 &= \frac{2}{d} + \frac{-\frac{8}{\pi} - 2}{d^2} + O\left(d^{\frac{-5}{2}}\right), \\
a_4 &= \frac{4}{\pi d} + \frac{32}{\pi^3 d^{\frac{3}{2}}} + \frac{8\left(-7\pi^3 - 8\pi^2 + 64\right)}{\pi^5 d^2} + O\left(d^{\frac{-5}{2}}\right), \\
a_5 &= -1 + \frac{\frac{8}{\pi^2} + 2 + \frac{8}{\pi}}{d} + - \frac{64(12-\pi)}{3\pi^4 d^{\frac{3}{2}} (-2+\pi)} + \\
&\quad - \frac{2\left(-128\pi^3 - 40\pi^4 - 224\pi^2 - 512\pi + 2560 + \pi^7 + 10\pi^6 + 52\pi^5\right)}{\pi^6 d^2 (-2+\pi)} + O\left(d^{\frac{-5}{2}}\right).
\end{aligned}$$

F.2 Eigenvalues

$\mathfrak{x}$-Representation	$\frac{(d-2)(d-3)}{2}$	$\frac{-2+\pi}{4\pi}$						
$\mathfrak{y}$-Representation	$\frac{(d-1)(d-4)}{2}$	$\frac{2+\pi}{4\pi}$						
Standard Representation	$d-2$	0	$\frac{-2+\pi}{4\pi}$	$\frac{-2+\pi}{2\pi}$	$\frac{1}{4}$	$\frac{2+\pi}{4\pi}$	$\frac{d}{4} + \frac{1}{2}$	
Trivial Representation	1	0	0	$\frac{-2+\pi}{2\pi}$	$\frac{1}{4}$	$\frac{d}{4} + \frac{-4+\pi+\pi^2}{-8\pi+2\pi^2}$	$\frac{d}{4} + \frac{1}{2}$	$\frac{d}{\pi} + \frac{-10\pi+8+\pi^2}{2\pi(-4+\pi)}$

F.3 Theorem 1, case 2b

$$
\begin{aligned}
a_1 &= -1 + \frac{2}{d} + -\frac{4\left(-16\pi^2 + (-2+\pi)\left(-6\pi^3 + 16\pi + 8\pi^2 + \pi^4\right) + 32\pi\right)}{\pi^3 d^2 (-2+\pi)^2} + O\left(d^{\frac{-5}{2}}\right), \\
a_2 &= \frac{2}{d} + \frac{-2 + \frac{8}{\pi}}{d^2} + O\left(d^{\frac{-5}{2}}\right), \\
a_3 &= -\frac{4\left(-32\pi + (-2+\pi)\left(-10\pi^2 - 8\pi + 3\pi^3\right) + 16\pi^2\right)}{\pi^3 d^2 (-2+\pi)^2} + O\left(d^{\frac{-5}{2}}\right), \\
a_4 &= \frac{2 - \frac{4}{\pi}}{d} + \frac{32(1-\pi)}{\pi^3 d^{\frac{3}{2}}} + \frac{-\frac{136}{\pi^2} - \frac{128}{\pi^3} - 2 - \frac{512}{\pi^5} + \frac{768}{\pi^4} + \frac{52}{\pi}}{d^2} + O\left(d^{\frac{-5}{2}}\right), \\
a_5 &= 1 + \frac{8(-1+\pi)}{\pi^2 d} + \\
&\quad -\frac{2\left(-90\pi^3 - 792\pi + \pi\sqrt{-160\pi^3 - 12\pi^5 - 192\pi + 64 + \pi^6 + 240\pi^2 + 60\pi^4} + 384 + 11\pi^4 + 468\pi^2\right)}{3\pi^4 d^{\frac{3}{2}} (-2+\pi)} \\
&\quad + \frac{h_4}{d^2} + O\left(d^{\frac{-5}{2}}\right).
\end{aligned}
$$

F.4 Eigenvalues

$\mathfrak{x}$-Representation	$\frac{(d-2)(d-3)}{2}$	$\frac{-2+\pi}{4\pi}$						
$\mathfrak{y}$-Representation	$\frac{(d-1)(d-4)}{2}$	$\frac{2+\pi}{4\pi}$						
Standard Representation	$d-2$	0	$\frac{-2+\pi}{4\pi}$	$\frac{-2+\pi}{2\pi}$	$\frac{1}{4}$	$\frac{2+\pi}{4\pi}$	$\frac{d}{4} + \frac{1}{2}$	
Trivial Representation	1	0	0	$\frac{-2+\pi}{2\pi}$	$\frac{1}{4}$	$\frac{d}{4} + \frac{-4+\pi+\pi^2}{-8\pi+2\pi^2}$	$\frac{d}{4} + \frac{1}{2}$	$\frac{d}{\pi} + \frac{-10\pi+8+\pi^2}{2\pi(-4+\pi)}$

F.5 Theorem 1, case 1b

$$
\begin{aligned}
a_1 &= -1 + \frac{2}{d} + -\frac{4\left(-16\pi^2 + (-2+\pi)\left(-6\pi^3 + 16\pi + 8\pi^2 + \pi^4\right) + 32\pi\right)}{\pi^3 d^2 (-2+\pi)^2} + O\left(d^{\frac{-5}{2}}\right), \\
a_2 &= \frac{2}{d} + \frac{-2 + \frac{8}{\pi}}{d^2} + O\left(d^{\frac{-5}{2}}\right).
\end{aligned}
$$

F.6 Eigenvalues

$\mathfrak{x}$-Representation	$\frac{(d-1)(d-2)}{2}$	$\frac{-2+\pi}{4\pi}$			
$\mathfrak{y}$-Representation	$\frac{d(d-3)}{2}$	$\frac{2+\pi}{4\pi}$			
Standard Representation	$d-1$	0	$\frac{-2+\pi}{2\pi}$	$\frac{1}{4}$	$\frac{d}{4} + \frac{1}{2}$
Trivial Representation	1	0	$\frac{d}{4} + \frac{-4+\pi+\pi^2}{2\pi(-4+\pi)}$	$\frac{d}{\pi} + \frac{-10\pi+8+\pi^2}{2\pi(-4+\pi)}$	

F.7 Theorem 1, case 3a

$$
\begin{aligned}
a_1 &= 1 + \frac{16}{\pi d^2} + O\Big(d^{\frac{-5}{2}}\Big), \\
a_2 &= -\frac{8}{\pi d^2} + O\Big(d^{\frac{-5}{2}}\Big), \\
a_3 &= \frac{2}{d} + \frac{2\left(-8\pi^2 - \pi^3 - 16 + 4\pi\right)}{\pi^2 d^2 \left(2+\pi\right)} + O\Big(d^{\frac{-5}{2}}\Big), \\
a_4 &= \frac{4}{\pi d} + \frac{32}{\pi^3 d^{\frac{3}{2}}} + \frac{4\left(-24\pi^4 - 160\pi^2 - \pi^5 + 256 + 192\pi + 28\pi^3\right)}{\pi^5 d^2 \left(2+\pi\right)} + O\Big(d^{\frac{-5}{2}}\Big), \\
a_5 &= -1 + \frac{\frac{8}{\pi^2} + 2 + \frac{8}{\pi}}{d} + -\frac{64\left(12-\pi\right)}{3\pi^4 d^{\frac{3}{2}}\left(-2+\pi\right)} \\
&\quad + \frac{2\left(-112\pi^5 - 3008\pi^2 - 240\pi^4 + 5120 + 2560\pi + \pi^8 + 8\sqrt{2}\pi^6 + 4\sqrt{2}\pi^7 + 672\pi^3 + 12\pi^7 + 104\pi^6\right)}{\pi^6 d^2\left(4-\pi^2\right)} \\
&\quad + O\Big(d^{\frac{-5}{2}}\Big), \\
a_6 &= \frac{2\left(-12\pi + 16 + \pi^3 + 4\pi^2\right)}{\pi^2 d\left(2+\pi\right)} + \frac{16\left(-24\pi^2 + 64 + 24\pi + \pi^4 + 4\pi^3\right)}{\pi^4 d^{\frac{3}{2}}\left(4+\pi^2+4\pi\right)} \\
&\quad + \frac{2\left(\begin{matrix}-184\pi^7 - 26\pi^8 - 3968\pi^3 - 8192\pi^2 - 16\sqrt{2}\pi^7 - 4\sqrt{2}\pi^8 - 112\pi^5 - \pi^9 \\ -16\sqrt{2}\pi^6 + 20480 + 24576\pi + 864\pi^4 + 112\pi^6\end{matrix}\right)}{\pi^6 d^2\left(8+\pi^3+12\pi+6\pi^2\right)} + O\Big(d^{\frac{-5}{2}}\Big).
\end{aligned}
$$

F.8 Eigenvalues

$\mathfrak{x}$-Representation	$\frac{(d-3)(d-4)}{2}$	$\frac{-2+\pi}{4\pi}$					
$\mathfrak{y}$-Representation	$\frac{(d-2)(d-5)}{2}$	$\frac{2+\pi}{4\pi}$					
Standard Representation $\mathfrak{s}_{d-2}$	$d-3$	0	$\frac{-2+\pi}{4\pi}$	$\frac{-2+\pi}{2\pi}$	$\frac{1}{4}$	$\frac{2+\pi}{4\pi}$	$\frac{d}{4}+\frac{1}{2}$
Standard Representation $\mathfrak{s}_2$	1	0	$\frac{-2+\pi}{4\pi}$	$\frac{-2+\pi}{2\pi}$	$\frac{1}{4}$	$\frac{d}{4}+\frac{1}{2}$	
Trivial Representation	1	0	0	$\frac{-2+\pi}{2\pi}$	$\frac{1}{4}$	$\frac{2+\pi}{4\pi}$	$\frac{d}{4}+\frac{-4+\pi+\pi^2}{-8\pi+2\pi^2}$
		$\frac{d}{4}+\frac{1}{2}$		$\frac{d}{\pi}+\frac{-10\pi+8+\pi^2}{2\pi(-4+\pi)}$			
Tensor Representation $\mathfrak{s}_{d-2}\otimes\mathfrak{s}_2$	$d-3$	$\frac{-2+\pi}{4\pi}$	$\frac{2+\pi}{4\pi}$				

F.9 Identity

$a_1 = 1, a_2 = 0.$

F.10 Eigenvalues

$\mathfrak{x}$-Representation	$\frac{(d-1)(d-2)}{2}$	$\frac{-2+\pi}{4\pi}$				
$\mathfrak{y}$-Representation	$\frac{d(d-3)}{2}$	$\frac{2+\pi}{4\pi}$				
Standard Representation	$d-1$	0	$\frac{-2+\pi}{2\pi}$	$\frac{1}{4}$	$\frac{d}{4}+\frac{1}{2}$	
Trivial Representation	1	0	$\frac{d}{4}+\frac{-4+\pi+\pi^2}{2\pi(-4+\pi)}$	$\frac{d}{\pi}+\frac{-10\pi+8+\pi^2}{2\pi(-4+\pi)}$		

F.11 Theorem 1, case 3b

$$a_1 = -1 + \frac{2}{d} + \frac{-4 + \frac{24}{\pi}}{d^2} + O\left(d^{\frac{-5}{2}}\right),$$

$$a_2 = \frac{2}{d} + \frac{-2 + \frac{12}{\pi}}{d^2} + O\left(d^{\frac{-5}{2}}\right),$$

$$a_3 = \frac{8\left(-4\pi - \pi^2 + 8\right)}{\pi^2 d^2 (2+\pi)} + O\left(d^{\frac{-5}{2}}\right),$$

$$a_4 = \frac{2 - \frac{4}{\pi}}{d} + \frac{32(1-\pi)}{\pi^3 d^{\frac{3}{2}}} + \frac{2\left(-344\pi^3 - \pi^6 - 512 + 4\pi^4 + 384\pi + 512\pi^2 + 26\pi^5\right)}{\pi^5 d^2 (2+\pi)} + O\left(d^{\frac{-5}{2}}\right),$$

$$a_5 = 1 + \frac{8(-1+\pi)}{\pi^2 d} + -\frac{8\left(-24\pi^3 - 200\pi + 96 + 3\pi^4 + 120\pi^2\right)}{3\pi^4 d^{\frac{3}{2}} (-2+\pi)}$$

$$+ \frac{2\left(-224\pi^6 - 6816\pi^3 - 496\pi^5 - 1088\pi^2 - \pi^8 - 5120 + 9728\pi + 52\pi^7 + 4112\pi^4\right)}{\pi^6 d^2 (4 - \pi^2)}$$

$$+ O\left(d^{\frac{-5}{2}}\right),$$

$$a_6 = \frac{4\left(-\pi^2 - 8 + 6\pi\right)}{\pi^2 d (2+\pi)} + \frac{16\left(-40\pi^2 - 40\pi - \pi^4 + 64 + 20\pi^3\right)}{\pi^4 d^{\frac{3}{2}} (4 + \pi^2 + 4\pi)}$$

$$+ \frac{4\left(-4816\pi^4 - 116\pi^7 - 4544\pi^3 - 10240 + 4096\pi + 5\pi^8 + 14848\pi^2 + 176\pi^6 + 1632\pi^5\right)}{\pi^6 d^2 (8 + \pi^3 + 12\pi + 6\pi^2)}$$

$$+ O\left(d^{\frac{-5}{2}}\right),$$

F.12 Eigenvalues

$\mathfrak{x}$-Representation	$\frac{(d-3)(d-4)}{2}$	$\frac{-2+\pi}{4\pi}$						
$\mathfrak{y}$-Representation	$\frac{(d-2)(d-5)}{2}$	$\frac{2+\pi}{4\pi}$						
Standard Representation $\mathfrak{s}_{d-2}$	$d-3$	0	$\frac{-2+\pi}{4\pi}$	$\frac{-2+\pi}{2\pi}$	$\frac{1}{4}$	$\frac{2+\pi}{4\pi}$	$\frac{d}{4} + \frac{1}{2}$	
Standard Representation $\mathfrak{s}_2$	1	0	$\frac{-2+\pi}{4\pi}$	$\frac{-2+\pi}{2\pi}$	$\frac{1}{4}$	$\frac{d}{4} + \frac{1}{2}$		
Trivial Representation	1	0	0	$\frac{-2+\pi}{2\pi}$	$\frac{1}{4}$	$\frac{2+\pi}{4\pi}$	$\frac{d}{4} + \frac{-4+\pi+\pi^2}{-8\pi+2\pi^2}$	
		$\frac{d}{4} + \frac{1}{2}$		$\frac{d}{\pi} + \frac{-10\pi+8+\pi^2}{2\pi(-4+\pi)}$				
Tensor Representation $\mathfrak{s}_{d-2} \otimes \mathfrak{s}_2$	$d-3$	$\frac{-2+\pi}{4\pi}$	$\frac{2+\pi}{4\pi}$					

F.13 Theorem 1, case 4

$$
\begin{aligned}
a_1 &= 1 + \frac{24}{\pi d^2} + O\Big(d^{\frac{-5}{2}}\Big), \\
a_2 &= -\frac{12}{\pi d^2} + O\Big(d^{\frac{-5}{2}}\Big), \\
a_3 &= \frac{2}{d} + \frac{2\left(-10\pi^2 - 32 - \pi^3 + 16\pi\right)}{\pi^2 d^2 \left(2+\pi\right)} + O\Big(d^{\frac{-5}{2}}\Big), \\
a_4 &= \frac{4}{\pi d} + \frac{32}{\pi^3 d^{\frac{3}{2}}} + \frac{8\left(-17\pi^4 - 144\pi^2 - \pi^5 + 128 + 128\pi + 50\pi^3\right)}{\pi^5 d^2 \left(2+\pi\right)} + O\Big(d^{\frac{-5}{2}}\Big), \\
a_5 &= -1 + \frac{\frac{8}{\pi^2} + 2 + \frac{8}{\pi}}{d} + -\frac{64\left(12-\pi\right)}{3\pi^4 d^{\frac{3}{2}}\left(-2+\pi\right)} \\
&\quad + \frac{2\left(-288\pi^5 - 5056\pi^2 - 272\pi^4 + 5120 + \pi^8 + 3584\pi + 8\sqrt{3}\pi^6 + 4\sqrt{3}\pi^7 + 16\pi^7 + 1824\pi^3 + 144\pi^6\right)}{\pi^6 d^2 \left(4-\pi^2\right)} \\
&\quad + O\Big(d^{\frac{-5}{2}}\Big), \\
a_6 &= \frac{2\left(-12\pi + 16 + \pi^3 + 4\pi^2\right)}{\pi^2 d \left(2+\pi\right)} + \frac{16\left(-24\pi^2 + 64 + 24\pi + \pi^4 + 4\pi^3\right)}{\pi^4 d^{\frac{3}{2}}\left(4 + \pi^2 + 4\pi\right)} \\
&\quad + \frac{2\left(\begin{matrix} -1920\pi^5 - 164\pi^7 - 28\pi^8 - 4992\pi^3 - 13824\pi^2 - 16\sqrt{3}\pi^7 - 4\sqrt{3}\pi^8 - \pi^9 \\ -16\sqrt{3}\pi^6 + 20480 + 28672\pi + 312\pi^6 + 4960\pi^4 \end{matrix}\right)}{\pi^6 d^2 \left(8 + \pi^3 + 12\pi + 6\pi^2\right)} + O\Big(d^{\frac{-5}{2}}\Big).
\end{aligned}
$$

F.14 Eigenvalues

$\mathfrak{x}_{d-3}$-Representation	$\frac{(d-5)(d-4)}{2}$	$\frac{-2+\pi}{4\pi}$						
$\mathfrak{x}_3$-Representation	1	$\frac{-2+\pi}{4\pi}$						
$\mathfrak{y}$-Representation	$\frac{(d-3)(d-6)}{2}$	$\frac{2+\pi}{4\pi}$						
Standard Representation $\mathfrak{s}_{d-3}$	$d-4$	0	$\frac{-2+\pi}{4\pi}$	$\frac{-2+\pi}{2\pi}$	$\frac{1}{4}$	$\frac{2+\pi}{4\pi}$	$\frac{d}{4}+\frac{1}{2}$	
Standard Representation $\mathfrak{s}_3$	2	0	$\frac{-2+\pi}{4\pi}$	$\frac{-2+\pi}{2\pi}$	$\frac{1}{4}$	$\frac{d}{4}+\frac{1}{2}$		
Trivial Representation	1	0 $\frac{d}{4}+\frac{1}{2}$	0	$\frac{-2+\pi}{2\pi}$ $\frac{d}{\pi}+\frac{-10\pi+8+\pi^2}{2\pi(-4+\pi)}$	$\frac{1}{4}$	$\frac{2+\pi}{4\pi}$	$\frac{d}{4}+\frac{-4+\pi+\pi^2}{-8\pi+2\pi^2}$	
Tensor Representation $\mathfrak{s}_{d-3}\otimes\mathfrak{s}_3$	$2d-8$	$\frac{-2+\pi}{4\pi}$	$\frac{2+\pi}{4\pi}$					