Locally private online change point detection (Supplementary materials)

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This file consists of all the technical details needed in the main text.

S.1 Proofs of main results

We first introduce some additional notation. For any integer pair $1 \leq s < t$, denote the un-smoothed and smoothed population CUSUM statistics as

$$D_{s,t} = \sup_{x \in \mathbb{R}^p} D_{s,t}(x) = \sqrt{\frac{s(t-s)}{t}} \sup_{x \in \mathbb{R}^p} |m_{1:s}(x) - m_{(s+1):t}(x)| \tag{S.1}$$

and

$$D^{(h)}_{s,t} = \sup_{x \in \mathbb{R}^p} D^{(h)}_{s,t}(x) = \sqrt{\frac{s(t-s)}{t}} \sup_{x \in \mathbb{R}^p} |m^{(h)}_{1:s}(x) - m^{(h)}_{(s+1):t}(x)|,$$

where

$$m_{s:t}(x) = \frac{1}{t-s+1} \sum_{i=s}^{t} m_i(x)$$

and

$$m^{(h)}_{s:t}(x) = \frac{1}{t-s+1} \sum_{i=s}^{t} \mathbb{E}\{Y_i | X_i \in A_{h,j}\}.$$

Proof of Theorem 3. For any integer pair (s,t), with $1 \leq s < t$, note that

$$\begin{aligned}
\left|\widetilde{D}_{s,t} - D_{s,t}\right| &= \left|\sqrt{\frac{s(t-s)}{t}} \max_{i=1}^{t} \left|\widetilde{m}_{1:s}(X_i) - \widetilde{m}_{(s+1):t}(X_i)\right| - D_{s,t}\right| \\
&\leq \sqrt{\frac{s(t-s)}{t}} \left|\max_{i=1}^{t} |\widetilde{m}_{1:s}(X_i) - \widetilde{m}_{(s+1):t}(X_i)| - \max_{i=1}^{t} |m^{(h)}_{1:s}(X_i) - m^{(h)}_{(s+1):t}(X_i)|\right| \\
&\quad + \sqrt{\frac{s(t-s)}{t}} \left|\max_{i=1}^{t} |m^{(h)}_{1:s}(X_i) - m^{(h)}_{(s+1):t}(X_i)| - \sup_{x \in \mathbb{R}^p} |m^{(h)}_{1:s}(x) - m^{(h)}_{(s+1):t}(x)|\right| \\
&\quad + \left|\sqrt{\frac{s(t-s)}{t}} \sup_{x \in \mathbb{R}^p} |m^{(h)}_{1:s}(x) - m^{(h)}_{(s+1):t}(x)| - D_{s,t}\right| \\
&= (I) + (II) + (III).
\end{aligned} \tag{S.2}$$

Step 1. In order to show (10) and (11), we focus on the integer pairs (s,t), with $1 \leq s < t \leq \Delta \leq \infty$. In this case, it follows from Lemma S.1 that $D_{s,t} = 0$, from Lemma S.2 that $(II) = 0$, from

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Lemma S.3 that $(III) = 0$, from Lemma S.5 that with probability $1-\gamma$,

$$(I) \leq 2\sqrt{\frac{s(t-s)}{t}} C_{\mathrm{Lip}} \sqrt{d}h + \frac{4\sigma}{\sqrt{c_{\min} h^d}} \sqrt{5\log(t) + \log(16/\gamma)}.$$

Therefore, the results (10) and (11) hold due to the choice of $b_{s,t}$ in (9).

Step 2. In order to show (12), due to (S.2), it suffices to show that

$$D_{\Delta,\Delta+\epsilon} - (I) - (II) - (III) > \widetilde{b}_{\Delta,\Delta+\epsilon}$$

with probability at least $1-\gamma$. In this case, it follows from Lemma S.1 that

$$D_{\Delta,\Delta+\epsilon} = \kappa\sqrt{\Delta}\sqrt{\frac{\epsilon}{\Delta+\epsilon}},$$

from Lemma S.2 that

$$\mathbb{P}\{(II) = 0\} \geq 1 - \frac{1}{c_{\min} h^d} \exp(-\Delta c_{\min} h^d),$$

for C_{SNR} large enough, from Lemma S.3 that

$$(III) \leq 2\sqrt{\frac{\Delta\epsilon}{\Delta+\epsilon}} C_{\mathrm{Lip}} \sqrt{d}h$$

and from Lemma S.5 that

$$(I) \leq 2\sqrt{\frac{\Delta\epsilon}{\Delta+\epsilon}} C_{\mathrm{Lip}} \sqrt{d}h + \frac{4\sigma}{\sqrt{c_{\min} h^d}} \sqrt{5\log(\Delta+\epsilon) + \log(32/\gamma)}$$

with probability at least $1-\gamma/2$. Note that, due to Assumption 5, $\Delta \geq \epsilon$. Combining the above statements, we thus have with probability at least $1-\gamma$ that

$$\begin{aligned}
& D_{\Delta,\Delta+\epsilon} - (I) - (II) - (III) - \widetilde{b}_{\Delta,\Delta+\epsilon} \\
\geq & \kappa\sqrt{\Delta}\sqrt{\frac{\epsilon}{\Delta+\epsilon}} - 2\sqrt{\frac{\Delta\epsilon}{\Delta+\epsilon}} C_{\mathrm{Lip}} \sqrt{d}h - \frac{4\sigma}{\sqrt{c_{\min} h^d}} \sqrt{5\log(\Delta+\epsilon) + \log(32/\gamma)} \\
& \quad - 2\sqrt{\frac{\Delta\epsilon}{\Delta+\epsilon}} C_{\mathrm{Lip}} \sqrt{d}h - 2\sqrt{\frac{\Delta\epsilon}{\Delta+\epsilon}} C_{\mathrm{Lip}} \sqrt{d}h - \frac{4\sigma}{\sqrt{c_{\min} h^d}} \sqrt{5\log(\Delta+\epsilon) + \log(32/\gamma)} \\
\geq & \kappa\sqrt{\Delta}\sqrt{\frac{\epsilon}{\Delta+\epsilon}} - 6\sqrt{\frac{\Delta\epsilon}{\Delta+\epsilon}} C_{\mathrm{Lip}} \sqrt{d}h - \frac{8\sigma}{\sqrt{c_{\min} h^d}} \sqrt{5\log(64\Delta/\gamma)} > 0,
\end{aligned}$$

where the second inequality follows from $\epsilon \leq \Delta$, and the last inequality holds with a large enough C_{SNR}. $\square$

Proof of Theorem 4. Throughout the proof, we will omit the use $\lceil\cdot\rceil$ or $\lfloor\cdot\rfloor$ notation, for the sake of simplicity.

Step 1. We let $f_1(\cdot) = m_\Delta$ and $f_2(\cdot) = m_{\Delta+1}$, which are the before and after change point mean functions respectively. To be specific, we let $\mathcal{X} = B(0, r_{\mathcal{X}})$, with

$$r_{\mathcal{X}} = \max\left\{\left(\frac{8\sigma^2\log(1/\gamma)}{\kappa^2}\right)^{1/d}, 2\kappa\right\}.$$

We let

$$f_2(x) = f_1(x) + \begin{cases} \kappa - \|x\|, & x \in B(0,\kappa) \\ 0, & x \in \mathcal{X} \setminus B(0,\kappa), \end{cases} \quad \text{and} \quad f_1(x) = 0, \forall x \in \mathcal{X}.$$

We let the distribution of X be uniform on $\mathcal{X}$ with density $p_X(x) = u = V_d^{-1} r_{\mathcal{X}}^{-d}$, for any $x \in \mathcal{X}$. We have that

$$\|f_2 - f_1\|_\infty = \kappa$$

and both f_j, $j = 1, 2$, are Lipschitz with constant upper bounded by 1.

Step 2. For any $n \in \mathbb{N}^*$, let P^n be the restriction of a distribution P to $\mathcal{F}_n$, i.e. the σ-field generated by the observations $\{(X_i, Y_i)\}_{i=1}^n$. For any $\nu \geq 1$ and $n \geq \nu$, we let

$$Z_{\nu,n} = \log\left(\frac{dP^n_{\kappa,\sigma,\nu}}{dP^n_{\kappa,\sigma,\infty}}\right) = \sum_{i=\nu+1}^{n} Z_i,$$

where $P_{\kappa,\sigma,\infty}$ indicates the joint distribution under which there is no change point and

$$Z_i = \log\left\{\frac{dP_{\kappa,\sigma,\nu}(X_i, Y_i)}{dP_{\kappa,\sigma,\nu}(X_i, Y_i)}\right\} = \frac{f_2(X_i) - f_1(X_i)}{\sigma^2}\left\{Y_i - \frac{f_1(X_i) + f_2(X_i)}{2}\right\}.$$

Step 2.1. For any $\nu \geq 1$, define the event

$$\mathcal{E}_\nu = \left\{\nu < T \leq \nu + \frac{\sigma^2}{\kappa^{2+d}}\log(1/\gamma),\ Z_{\nu,T} < \frac{3}{4}\log\left(\frac{1}{\gamma}\right)\right\}.$$

Then we have

$$\begin{aligned}\mathbb{P}_{\kappa,\sigma,\nu}(\mathcal{E}_\nu) &= \int_{\mathcal{E}_\nu} \exp\left(Z_{\nu,T}\right)\, dP_{\kappa,\sigma,\infty} \leq \gamma^{-3/4}\mathbb{P}_{\kappa,\sigma,\infty}(\mathcal{E}_\nu) \\ &\leq \gamma^{-3/4}\mathbb{P}_{\kappa,\sigma,\infty}\left\{\nu < T \leq \nu + \frac{\sigma^2}{\kappa^{2+d}}\log(1/\gamma)\right\} \leq \gamma^{-3/4}\gamma = \gamma^{1/4},\end{aligned} \tag{S.3}$$

where the last inequality follows from the definition of $\mathcal{D}(\gamma)$.

Step 2.2. For any $\nu \geq 1$ and $T \in \mathcal{D}(\gamma)$, since $\{T \geq \nu\} \in \mathcal{F}_{\nu-1}$, we have that

$$\begin{aligned}&\mathbb{P}_{\kappa,\sigma,\nu}\left\{\nu < T \leq \nu + \frac{\sigma^2}{\kappa^{2+d}}\log(1/\gamma),\ Z_{\nu,T} \geq \frac{3}{4}\log\left(\frac{1}{\gamma}\right)\Big| T > \nu\right\} \\ \leq& \operatorname{ess\,sup}\mathbb{P}_{\kappa,\sigma,\nu}\left\{\max_{1\leq t\leq \frac{\sigma^2}{\kappa^{2+d}}\log\left(\frac{1}{\gamma}\right)} Z_{\nu,\nu+t} \geq \frac{3}{4}\log\left(\frac{1}{\gamma}\right)\Big|(X_i, Y_i)_{i=1}^\nu\right\} \\ \leq& \frac{\sigma^2}{\kappa^{2+d}}\log\left(\frac{1}{\gamma}\right)\max_{1\leq t\leq \frac{\sigma^2}{\kappa^{2+d}}\log\left(\frac{1}{\gamma}\right)} \mathbb{E}_X\mathbb{P}_\varepsilon\left\{Z_{\nu,\nu+t} \geq \frac{3}{4}\log\left(\frac{1}{\gamma}\right)\Big|(X_i, Y_i)_{i=1}^\nu\right\} \\ =& \frac{\sigma^2}{\kappa^{2+d}}\log\left(\frac{1}{\gamma}\right)\left\{\max_{1\leq t\leq \frac{\sigma^2}{\kappa^{2+d}}\log\left(\frac{1}{\gamma}\right)} g(t) + \max_{1\leq t\leq \frac{\sigma^2}{\kappa^{2+d}}\log\left(\frac{1}{\gamma}\right)} h(t)\right\} = (I) + (II),\end{aligned}$$

where

$$g(t) = \mathbb{E}_X\mathbb{P}_\varepsilon\left\{\left[Z_{\nu,\nu+t} \geq \frac{3}{4}\log\left(\frac{1}{\gamma}\right)\right] \cap \left[\sum_{i=\nu+1}^{\nu+t}\frac{\{f_2(X_i) - f_1(X_i)\}^2}{2\sigma^2} \leq \frac{1}{4}\log\left(\frac{1}{\gamma}\right)\right]\right\}$$

and

$$h(t) = \mathbb{E}_X\mathbb{P}_\varepsilon\left\{\left[Z_{\nu,\nu+t} \geq \frac{3}{4}\log\left(\frac{1}{\gamma}\right)\right] \cap \left[\sum_{i=\nu+1}^{\nu+t}\frac{\{f_2(X_i) - f_1(X_i)\}^2}{2\sigma^2} > \frac{1}{4}\log\left(\frac{1}{\gamma}\right)\right]\right\}.$$

Step 2.2.1. We first deal with (II). As for $h(t)$, we have that

$$\begin{aligned}h(t) &\leq \mathbb{P}_X\left\{\sum_{i=\nu+1}^{\nu+t}\frac{\{f_2(X_i) - f_1(X_i)\}^2}{2\sigma^2} > \frac{1}{4}\log\left(\frac{1}{\gamma}\right)\right\} \\ &\leq \mathbb{P}_X\left\{\sum_{i=\nu+1}^{\nu+\frac{\sigma^2}{\kappa^{2+d}}\log\left(\frac{1}{\gamma}\right)}\frac{\{f_2(X_i) - f_1(X_i)\}^2}{2\sigma^2} > \frac{1}{4}\log\left(\frac{1}{\gamma}\right)\right\}.\end{aligned}$$

If

$$\frac{1}{2\kappa^d}\log\left(\frac{1}{\gamma}\right) \leq \frac{1}{4}\log\left(\frac{1}{\gamma}\right),$$

i.e. $\kappa \geq 2^{1/d}$, then $h(t) = 0$. Otherwise,

$$h(t) \leq V_d^t \kappa^{dt} u^t \leq V_d \kappa^d u,$$

where the last inequality is due to the fact that $uV_d\kappa^d < 1$.

Then we have that

$$\begin{aligned}
(II) &= \frac{\sigma^2}{\kappa^{2+d}}\log\left(\frac{1}{\gamma}\right) \max_{1\leq t\leq \frac{\sigma^2}{\kappa^{2+d}}\log\left(\frac{1}{\gamma}\right)} h(t) = \frac{\sigma^2}{\kappa^{2+d}}\log\left(\frac{1}{\gamma}\right) V_d\kappa^d u \\
&= \frac{\sigma^2}{\kappa^2}\log\left(\frac{1}{\gamma}\right)\min\left\{\frac{\kappa^2}{8\sigma^2\log(1/\gamma)},\, \frac{1}{2^d\kappa^d}\right\} \leq 1/8. \qquad \text{(S.4)}
\end{aligned}$$

Step 2.2.2. We then deal with (I). As for $g(t)$, we have that

$$\begin{aligned}
g(t) &= \mathbb{E}_X\mathbb{P}_\varepsilon\left\{\left[Z_{\nu,\nu+t} \geq \frac{3}{4}\log\left(\frac{1}{\gamma}\right)\right] \cap \left[\sum_{i=\nu+1}^{\nu+t}\frac{\{f_2(X_i)-f_1(X_i)\}^2}{2\sigma^2} \leq \frac{1}{4}\log\left(\frac{1}{\gamma}\right)\right]\right\} \\
&\leq \mathbb{E}_X\mathbb{P}_\varepsilon\left\{Z_{\nu,\nu+t} - \sum_{i=\nu+1}^{\nu+t}\frac{\{f_2(X_i)-f_1(X_i)\}^2}{2\sigma^2} \geq \frac{1}{2}\log\left(\frac{1}{\gamma}\right)\right\} \\
&\leq \mathbb{E}_X\left[\exp\left\{-\frac{(1/2)^2\log^2(1/\gamma)}{2\sum_{i=\nu+1}^{\nu+t}\frac{\{f_2(X_i)-f_1(X_i)\}^2}{\sigma^2}}\right\}\right] \\
&= \int\cdots\int_{B(0,\kappa)^{\otimes t}} \exp\left\{-\frac{(1/2)^2\log^2(1/\gamma)}{2\sum_{i=1}^{t}\frac{\{f_2(x_i)-f_1(x_i)\}^2}{\sigma^2}}\right\} u^t\, dx_1\cdots dx_t \\
&= \int\cdots\int_{B(0,\kappa)^{\otimes t}} \exp\left\{-\frac{(1/2)^2\log^2(1/\gamma)}{2\sum_{i=1}^{t}\frac{(\kappa-\|x_i\|)^2}{\sigma^2}}\right\} u^t\, dx_1\cdots dx_t \\
&\leq u^t\int\cdots\int_{B(0,\kappa)^{\otimes t}} \exp\left\{-\frac{(1/2)^2\log^2(1/\gamma)}{2t\frac{\kappa^2}{\sigma^2}}\right\} dx_1\cdots dx_t \\
&= \exp\left\{-\frac{(1/2)^2\log^2(1/\gamma)}{2t\frac{\kappa^2}{\sigma^2}}\right\}\left(uV_d\kappa^d\right)^t.
\end{aligned}$$

Therefore

$$\begin{aligned}
(I) &= \frac{\sigma^2}{\kappa^{2+d}}\log\left(\frac{1}{\gamma}\right) \max_{1\leq t\leq \frac{\sigma^2}{\kappa^{2+d}}\log\left(\frac{1}{\gamma}\right)} g(t) \\
&\leq \frac{\sigma^2}{\kappa^{2+d}}\log\left(\frac{1}{\gamma}\right)\exp\left\{-\frac{(1/2)^2\log^2(1/\gamma)}{2\frac{\sigma^2}{\kappa^{2+d}}\log\left(\frac{1}{\gamma}\right)\frac{\kappa^2}{\sigma^2}}\right\} uV_d\kappa^d \\
&\leq (1/8)\exp\left\{-\frac{\kappa^d\log(1/\gamma)}{8}\right\} = (1/8)\gamma^{\kappa^d} < 1/8,
\end{aligned}$$

where the second inequality is due to (S.4).

Step 2.2.2. Combining the previous two steps, we have that

$$\mathbb{P}_{\kappa,\sigma,\nu}\left\{\nu < T \leq \nu + \frac{\sigma^2}{\kappa^{2+d}}\log(1/\gamma),\, Z_{\nu,T} \geq \frac{3}{4}\log\left(\frac{1}{\gamma}\right)\Big| T > \nu\right\} < 1/4. \qquad \text{(S.5)}$$

Step 3. Combining (S.3) and (S.5), we have that

$$\mathbb{P}_{\kappa,\sigma,\nu}\left\{\nu < T \le \nu + \frac{\sigma^2}{\kappa^{2+d}}\log(1/\gamma)\right\} \le \gamma^{1/4} + 1/4.$$

Since the upper bound in the above display is independent of ν, we have that

$$\sup_{\nu\ge 1}\mathbb{P}_{\kappa,\sigma,\nu}\left\{\nu < T \le \nu + \frac{\sigma^2}{\kappa^{2+d}}\log(1/\gamma)\right\} \le \gamma^{1/4} + 1/4.$$

Therefore, for any change point time Δ, we have that

$$\begin{aligned}
&\mathbb{E}_{\kappa,\sigma,\Delta}\{(T-\Delta)_+\} \ge \frac{\sigma^2}{\kappa^{2+d}}\log(1/\gamma)\mathbb{P}_{\kappa,\sigma,\nu}\left\{T-\nu > \frac{\sigma^2}{\kappa^{2+d}}\log(1/\gamma)\right\}\\
=&\frac{\sigma^2}{\kappa^{2+d}}\log(1/\gamma)\left[\mathbb{P}_{\kappa,\sigma,\nu}\left\{T>\nu\right\} - \mathbb{P}_{\kappa,\sigma,\nu}\left\{\nu < T \le \nu + \frac{\sigma^2}{\kappa^{2+d}}\log(1/\gamma)\right\}\right]\\
\ge&\frac{\sigma^2}{\kappa^{2+d}}\log(1/\gamma)(1-\gamma-\gamma^{1/4}-1/4) \ge \frac{\sigma^2}{2\kappa^{2+d}}\log(1/\gamma),
\end{aligned}$$

where the last inequality holds when $\gamma + \gamma^{1/4} < 1/4$. □

Proof of Theorem 1. In the case that

$$\frac{s(t-s)}{t}c_{\min}^2 h^{2d}\alpha^2 \ge 64\log\Big(\frac{72t^3}{\gamma c_{\min} h^d}\Big),$$

we start by writing

$$\begin{aligned}
|\widehat{D}_{s,t} - D_{s,t}| \le& \sqrt{\frac{s(t-s)}{t}}\left|\sup_{x\in\mathbb{R}^d}|\widehat{m}_{1:s}(x) - \widehat{m}_{(s+1):t}(x)| - \sup_{x\in\mathbb{R}^d}|m_{1:s}^{(h)}(x) - m_{(s+1):t}^{(h)}(x)|\right|\\
&+\left|\sqrt{\frac{s(t-s)}{t}}\sup_{x\in\mathbb{R}^d}\left|m_{1:s}^{(h)}(x) - m_{(s+1):t}^{(h)}(x)\right| - D_{s,t}\right|\\
=&(I') + (III).
\end{aligned}$$

Here (III) is the same term as appears in the proof of Theorem 3, and we do not need to bound it again here. The term (I') roughly corresponds to the non-private (I) and (II), but new techniques must be applied. Instead of using Lemma S.5 we use Lemma S.7, and we also need to bound a truncation error, but otherwise the proof is very similar.

For regression functions m we must bound the truncation error

$$\max_j \frac{1}{\mu(A_{h,j})}\left|\int_{A_{h,j}}\{m(x) - \mathbb{E}([Y]_{-M}^M|X=x)\}\mu(dx)\right|,$$

and this changes the overall proof by just adding some bias. We use the facts that $\sup_x |m(x)| \le M_0$ and $Y|X=x$ is σ-subgaussian for all x. For a σ-subgaussian random variable Z and $t \ge \sigma\sqrt{\log(4)}$ we have that (using Exercise 2.3 of [3])

$$\mathbb{E}(|Z|\mathbb{1}_{|Z|\ge t}) \le t\inf_{k\in\mathbb{N}}\frac{\mathbb{E}(|Z|^k)}{t^k} \le t\inf_{\lambda>0}\frac{\mathbb{E}(e^{\lambda|Z|})}{e^{\lambda t}} \le 2t\inf_{\lambda\in\mathbb{R}}\frac{\mathbb{E}(e^{\lambda Z})}{e^{\lambda t}} \le 2t\inf_{\lambda\in\mathbb{R}}e^{-\lambda t+\lambda^2\sigma^2/2} = 2te^{-\frac{t^2}{2\sigma^2}}.$$

When $M \ge M_0 + \sigma\sqrt{2\log(2+\sigma/h) + \log\log(2+\sigma/h)}$ we therefore have

$$\begin{aligned}
&\max_j \frac{1}{\mu(A_{h,j})}\left|\int_{A_{h,j}}\{m(x) - \mathbb{E}([Y]_{-M}^M|X=x)\}\mu(dx)\right|\\
&\le \max_j \frac{1}{\mu(A_{h,j})}\int_{A_{h,j}}\mathbb{E}\{(|Y|-M)\mathbb{1}_{|Y|\ge M}|X=x\}\mu(dx)\\
&\le \max_j \frac{1}{\mu(A_{h,j})}\int_{A_{h,j}}\mathbb{E}\{(|Y|-m(x))\mathbb{1}_{|Y-m(x)|\ge M-M_0}|X=x\}\mu(dx)
\end{aligned}$$

$$
\begin{aligned}
&\leq 2(M-M_0)\exp\left(-\frac{(M-M_0)^2}{2\sigma^2}\right)\\
&\leq 2\frac{\sigma\sqrt{2\log(2+\sigma/h)+\log\log(2+\sigma/h)}}{(2+\sigma/h)\sqrt{\log(2+\sigma/h)}} \leq 2\sqrt{2}h,
\end{aligned}
$$

which is the same order as the bias. We see that we have

$$
\mathbb{P}\Big(|\widehat{D}_{s,t}-D_{s,t}| \geq b_{s,t}\Big) \leq \gamma/(2t^3). \tag{S.6}
$$

On the other hand, in the case that

$$
\frac{s(t-s)}{t}c_{\min}^2 h^{2d}\alpha^2 < 64\log\Big(\frac{72t^3}{\gamma c_{\min}h^d}\Big),
$$

we will never flag a change, and there is nothing to prove.

In the setting of Assumption 2 we see from (S.6) and a union bound argument that we have

$$
\mathbb{P}\Big(\max_{t\in\mathbb{N}^*}\max_{1\leq s\leq t}(\widehat{D}_{s,t}-b_{s,t})\geq 0\Big) \leq \sum_{t=1}^{\infty} t\frac{\gamma}{2t^3} = \gamma\frac{\pi^2}{12} \leq \gamma.
$$

Similarly, in the setting of Assumption 3, we have

$$
\mathbb{P}\Big(\max_{1\leq t\leq \Delta}\max_{1\leq s\leq t}(\widehat{D}_{s,t}-b_{s,t})\geq 0\Big) \leq \sum_{t=1}^{\Delta} t\frac{\gamma}{2t^3} \leq \gamma.
$$

We have that

$$
\frac{\Delta\epsilon}{\Delta+\epsilon}c_{\min}^2 h^{2d}\alpha^2 \geq 64\log\Big(\frac{72t^3}{\gamma c_{\min}h^d}\Big)
$$

when C_ϵ and C_{SNR} are chosen large enough. Thus, by (S.6) and Lemma S.1, with probability at least $1-\gamma$, we have

$$
\begin{aligned}
\widehat{D}_{\Delta,\Delta+\epsilon} - b_{\Delta,\Delta+\epsilon} &\geq D_{\Delta,\Delta+\epsilon} - 2b_{\Delta,\Delta+\epsilon}\\
&\geq \kappa\sqrt{\frac{\Delta\epsilon}{\Delta+\epsilon}} - 4\sqrt{\frac{\Delta\epsilon}{\Delta+\epsilon}}(C_{\mathrm{Lip}}\sqrt{d}+2\sqrt{2})h - \frac{2M}{c_{\min}h^d\alpha}\sqrt{\log\Big(\frac{72t^3}{\gamma c_{\min}h^d}\Big)} > 0,
\end{aligned}
$$

where the final inequality holds when κ/h is above a constant threshold and C_{SNR} is large enough.

$\square$

Proof of Theorem 2. Throughout the proof, we will omit the use $\lceil\cdot\rceil$ or $\lfloor\cdot\rfloor$ notation, for the sake of simplicity.

Step 1. Let Q be any sequentially interactive privacy mechanism, with output in some space $\mathcal{W}$, whose density satisfies

$$
\frac{q_i(w|x,y,w_{i-1},\ldots,w_1)}{q_i(w|x',y',w_{i-1},\ldots,w_1)} \leq e^\alpha, \quad \forall w,w_1,\ldots,w_{i-1}\in\mathcal{W},\ x,x'\in\mathbb{R}^d,\ y,y'\in\mathbb{R}.
$$

Let $f_1(\cdot)=m_\Delta$ and $f_2(\cdot)=m_{\Delta+1}$, which are the before and after change point mean functions respectively. Let $P_X(\cdot)$ be the distribution of X, staying the same before and after the change point. Let $Y_i = m_i(X)+\mathrm{Unif}[-\sigma,\sigma]$, for any $i\in\mathbb{N}^*$. Let $P_{k|X}$, $k=1,2$, denote the distribution of Y given X, before and after the change point. Let $h_{i,k}(w|w_{i-1},\ldots,w_1) = \int q(w|x,y,w_{i-1},\ldots,w_1)\,dP_X(x)dP_{k|X}(y|x)$. Let $\mathcal{X}=B(0,r_\mathcal{X})$, with

$$
r_\mathcal{X} = \left(\frac{2(e^\alpha-1)}{\alpha}\right)^{1/d}. \tag{S.7}
$$

We let

$$
f_2(x) = f_1(x) + \begin{cases}\kappa-\|x\|, & x\in B(0,\kappa)\\ 0, & x\in\mathcal{X}\setminus B(0,\kappa),\end{cases} \quad \text{and} \quad f_1(x)=0,\ \forall x\in\mathcal{X}.
$$

We let the distribution of X be uniform on $\mathcal{X}$ with density $p_X(x) = u = V_d^{-1} r_{\mathcal{X}}^{-d}$, for any $x \in \mathcal{X}$. We have that

$$\|f_2 - f_1\|_\infty = \kappa$$

and both f_j, $j = 1, 2$, are Lipschitz with constant upper bounded by 1.

Step 2. For any $n \in \mathbb{N}^*$, let P^n be the restriction of a distribution P to $\mathcal{F}_n$, i.e. the σ-field generated by the observations $\{W_i\}_{i=1}^n \subset \mathcal{W}$. For any $\nu \geq 1$ and $n \geq \nu$, we let

$$Z_{\nu,n} = \log\left(\frac{dP^n_{\kappa,\sigma,\nu}}{dP^n_{\kappa,\sigma,\infty}}\right) = \sum_{i=\nu+1}^{n} Z_i = \sum_{i=\nu+1}^{n} \log\left\{\frac{h_{i,2}(W_i|W_{i-1},\ldots,W_1)}{h_{i,1}(W_i|W_{i-1},\ldots,W_1)}\right\},$$

where $P_{\kappa,\sigma,\infty}$ indicates the joint distribution under which there is no change point.

It follows from Lemma 1 in [2, Supplementary material], that we have

$$|Z_i| \leq \min\{2,\, e^{\alpha}\}(e^{\alpha} - 1) d_{\mathrm{TV}}(P_1, P_2), \quad i \in \{\nu+1, \ldots, n\}.$$

where P_1 and P_2 are the joint distributions of (X, Y) before and after the change point. Moreover, by calculations around Lemma 1 in [2, Supplementary material] we have

$$0 \leq \int h_{i,2}(w|w_{i-1},\ldots,w_1) \log \frac{h_{i,2}(w|w_{i-1},\ldots,w_1)}{h_{i,1}(w|w_{i-1},\ldots,w_1)}\, du \leq \min(4, e^{2\alpha})(e^{\alpha}-1)^2 d_{\mathrm{TV}}(P_1,P_2)^2$$

for all $w_1, \ldots, w_{i-1}$. Since

$$d_{\mathrm{TV}}(P_1, P_2) = \frac{1}{2\sigma}\int_{B(0,\kappa)} (\kappa - \|x\|) p_X(x)\, dx \leq \frac{\kappa}{2\sigma} V_d \kappa^d u,$$

we have that for any $i > \nu$, $|Z_i| \leq \min\{2,\, e^{\alpha}\}(e^{\alpha} - 1)\frac{\kappa}{2\sigma} V_d \kappa^d u \leq \alpha\kappa^{d+1}/(2\sigma)$, and $\mathbb{E}(Z_i|W_{i-1},\ldots,W_1) \leq \alpha^2\kappa^{2d+2}/(4\sigma^2)$ almost surely.

Step 2.1. For any $\nu \geq 1$, define the event

$$\mathcal{E}_\nu = \left\{\nu < T \leq \nu + \frac{\sigma^2\log(1/\gamma)}{\kappa^{2+2d}\alpha^2},\, Z_{\nu,T} < \frac{3}{4}\log\left(\frac{1}{\gamma}\right)\right\}.$$

Then we have

$$\begin{aligned}
\mathbb{P}_{\kappa,\sigma,\nu}(\mathcal{E}_\nu) &= \int_{\mathcal{E}_\nu} \exp\left(Z_{\nu,T}\right)\, dP_{\kappa,\sigma,\infty} \leq \gamma^{-3/4}\mathbb{P}_{\kappa,\sigma,\infty}(\mathcal{E}_\nu) \\
&\leq \gamma^{-3/4}\mathbb{P}_{\kappa,\sigma,\infty}\left\{\nu < T \leq \nu + \frac{\sigma^2\log(1/\gamma)}{\kappa^{2+2d}\alpha^2}\right\} \leq \gamma^{-3/4}\gamma = \gamma^{1/4}, \qquad \text{(S.8)}
\end{aligned}$$

where the last inequality follows from the definition of $\mathcal{D}(\gamma)$.

Step 2.2. For any $\nu \geq 1$ and $T \in \mathcal{D}(\gamma)$, since $\{T \geq \nu\} \in \mathcal{F}_{\nu-1}$, we have that

$$\begin{aligned}
&\mathbb{P}_{\kappa,\sigma,\nu}\left\{\nu < T \leq \nu + \frac{\sigma^2\log(1/\gamma)}{\kappa^{2+2d}\alpha^2},\, Z_{\nu,T} \geq \frac{3}{4}\log\left(\frac{1}{\gamma}\right) \Big| T > \nu\right\} \\
\leq &\operatorname{ess\,sup} \mathbb{P}_{\kappa,\sigma,\nu}\left\{\max_{1\leq t\leq \frac{\sigma^2\log(1/\gamma)}{\kappa^{2+2d}\alpha^2}} Z_{\nu,\nu+t} \geq \frac{3}{4}\log\left(\frac{1}{\gamma}\right) \Big| (W_i)_{i=1}^{\nu}\right\} \\
\leq &\frac{\sigma^2\log(1/\gamma)}{\kappa^{2+2d}\alpha^2} \max_{1\leq t\leq \frac{\sigma^2\log(1/\gamma)}{\kappa^{2+2d}\alpha^2}} \mathbb{P}_{\kappa,\sigma,\nu}\left\{Z_{\nu,\nu+t} \geq \frac{3}{4}\log\left(\frac{1}{\gamma}\right) \Big| (W_i)_{i=1}^{\nu}\right\}. \\
= &\frac{\sigma^2\log(1/\gamma)}{\kappa^{2+2d}\alpha^2} \max_{1\leq t\leq \frac{\sigma^2\log(1/\gamma)}{\kappa^{2+2d}\alpha^2}} g(t),
\end{aligned}$$

where

$$g(t) = \mathbb{P}_{\kappa,\sigma,\nu}\left\{Z_{\nu,\nu+t} \geq \frac{3}{4}\log\left(\frac{1}{\gamma}\right)\right\}.$$

As for $g(t)$, we have that

$$
\begin{aligned}
g(t) &\leq \mathbb{P}_{\kappa,\sigma,\nu}\left\{\sum_{i=\nu+1}^{\nu+t}\{Z_i - \mathbb{E}(Z_i|W_{i-1},\ldots,W_1)\} \geq \frac{3}{4}\log\left(\frac{1}{\gamma}\right) - \frac{\sigma^2\log(1/\gamma)}{\kappa^{2+2d}\alpha^2}\frac{\alpha^2\kappa^{2d+2}}{4\sigma^2}\right\} \\
&\leq \mathbb{P}_{\kappa,\sigma,\nu}\left\{\sum_{i=\nu+1}^{\nu+t}\{Z_i - \mathbb{E}(Z_i|W_{i-1},\ldots,W_1)\} \geq \frac{1}{2}\log\left(\frac{1}{\gamma}\right)\right\} \\
&\leq \exp\left\{-\frac{\log^2(1/\gamma)}{\frac{\sigma^2\log(1/\gamma)}{\kappa^{2+2d}\alpha^2}\alpha^2\kappa^{2d+2}/\sigma^2}\right\} = \gamma,
\end{aligned}
$$

where the second inequality is due the Azuma–Hoeffding inequality [e.g. 3, Corollary 2.20]. Therefore

$$
\mathbb{P}_{\kappa,\sigma,\nu}\left\{\nu < T \leq \nu + \frac{\sigma^2}{\kappa^{2+d}}\log(1/\gamma),\ Z_{\nu,T} \geq \frac{3}{4}\log\left(\frac{1}{\gamma}\right)\Big|T>\nu\right\} \leq \frac{\sigma^2\log(1/\gamma)}{\kappa^{2+2d}\alpha^2}\gamma \leq \gamma^{1/4}.
$$

Step 3. Combining the above we have that

$$
\mathbb{P}_{\kappa,\sigma,\nu}\left\{\nu < T \leq \nu + \frac{\sigma^2}{\kappa^{2+d}}\log(1/\gamma),\ Z_{\nu,T} \geq \frac{3}{4}\log\left(\frac{1}{\gamma}\right)\Big|T>\nu\right\} < 2\gamma^{1/4}.
$$

Since the upper bound in the above display is independent of ν, we have that

$$
\sup_{\nu\geq 1}\mathbb{P}_{\kappa,\sigma,\nu}\left\{\nu < T \leq \nu + \frac{\sigma^2}{\kappa^{2+d}}\log(1/\gamma)\right\} \leq 2\gamma^{1/4}.
$$

Therefore, for any change point time Δ, we have that

$$
\begin{aligned}
&\mathbb{E}_{\kappa,\sigma,\Delta}\{(T-\Delta)_+\} \geq \frac{\sigma^2\log(1/\gamma)}{\kappa^{2+2d}\alpha^2}\mathbb{P}_{\kappa,\sigma,\nu}\left\{T-\nu > \frac{\sigma^2\log(1/\gamma)}{\kappa^{2+2d}\alpha^2}\right\} \\
=&\frac{\sigma^2\log(1/\gamma)}{\kappa^{2+2d}\alpha^2}\left[\mathbb{P}_{\kappa,\sigma,\nu}\left\{T>\nu\right\} - \mathbb{P}_{\kappa,\sigma,\nu}\left\{\nu < T \leq \nu + \frac{\sigma^2\log(1/\gamma)}{\kappa^{2+2d}\alpha^2}\right\}\right] \\
\geq&\frac{\sigma^2\log(1/\gamma)}{\kappa^{2+2d}\alpha^2}(1-\gamma-2\gamma^{1/4}) \geq \frac{\sigma^2\log(1/\gamma)}{2\kappa^{2+2d}\alpha^2}.
\end{aligned}
$$

□

S.2 Auxiliary lemmas

S.2.1 Population quantities

Lemma S.1. *For $D_{s,t}$ defined in* (S.1)*, it holds that*

$$
D_{s,t} = \begin{cases} 0, & 1 \leq s < t \leq \Delta, \\ \kappa\sqrt{\Delta}\sqrt{\frac{t-\Delta}{t}}, & s = \Delta < t. \end{cases}
$$

Proof. When $1 \leq s < t \leq \Delta$, by definition, we have that $D_{s,t} = 0$.

When $s = \Delta < t$, let $m_\Delta = f_1$ and $m_{\Delta+1} = f_2$. Note that

$$
D_{\Delta,t} = \sqrt{\frac{\Delta(t-\Delta)}{t}}\|f_1 - f_2\|_\infty = \kappa\sqrt{\Delta}\sqrt{\frac{t-\Delta}{t}},
$$

where the last identity follows from Assumption 3. □

Lemma S.2. *When $1 \leq s < t \leq \Delta$, it holds that*

$$
\mathbb{P}\left\{\left|\max_{i=1}^{t} D_{s,t}^{(h)}(X_i) - D_{s,t}^{(h)}\right| = 0\right\} = 1.
$$

When $s = \Delta < t$, it holds that

$$
\mathbb{P}\left\{\left|\max_{i=1}^{t} D_{s,t}^{(h)}(X_i) - D_{s,t}^{(h)}\right| > 0\right\} \leq \frac{1}{c_{\min}h^d}\exp(-\Delta c_{\min}h^d)
$$

Proof. **Case 1.** When $1 \leq s < t \leq \Delta$, by definition, we have that

$$D_{s,t}^{(h)}(x) = 0, \quad x \in \mathbb{R}^d.$$

Then we have that

$$\mathbb{P}\left\{\left|\max_{i=1}^{t} D_{s,t}^{(h)}(X_i) - D_{s,t}^{(h)}\right| = 0\right\} = 1.$$

Case 2. The only way that we can have $\max_{i=1}^{t} D_{s,t}^{(h)}(X_i) \neq D_{s,t}^{(h)}$ is if there exists some $A_{h,j}$ with no observations falling within it. We have that

$$1 = \sum_j \mu(A_{h,j}) \geq c_{\min} h^d N_h$$

Thus, when $s = \Delta < t$, using a union bound, it holds that

$$\begin{aligned}
\mathbb{P}\left\{\left|\max_{i=1}^{t} D_{s,t}^{(h)}(X_i) - D_{s,t}^{(h)}\right| > 0\right\} &\leq \mathbb{P}\left\{\sum_{i=1}^{t} \mathbb{1}_{\{X_i \in A_{h,j}\}} = 0 \text{ for some } j\right\} \\
&\leq \frac{1}{c_{\min} h^d} \max_j \{1 - \mu(A_{h,j})\}^t \\
&\leq \frac{1}{c_{\min} h^d} \exp(-\Delta c_{\min} h^d),
\end{aligned}$$

as required.

□

Lemma S.3. *For any integer pairs (s,t), $1 \leq s < t \leq \Delta$, it holds that $|D_{s,t}^{(h)} - D_{s,t}| = 0$. When $s = \Delta < t$, it holds that*

$$|D_{s,t}^{(h)} - D_{s,t}| \leq 2\sqrt{\frac{s(t-s)}{t}} C_{\text{Lip}} \sqrt{d} h.$$

Proof. For any integer pairs (s,t), $1 \leq s < t \leq \Delta$, it holds that

$$m_{1:s}^{(h)}(x) = m_{(s+1):t}^{(h)}(x) \quad \text{and} \quad m_{1:s}(x) = m_{(s+1):t}(x), \quad \forall x \in \mathbb{R}^d,$$

which implies that $|D_{s,t}^{(h)} - D_{s,t}| = 0$.

When $s = \Delta < t$, it holds that

$$\begin{aligned}
&|D_{\Delta,t}^{(h)} - D_{\Delta,t}| \\
=&\sqrt{\frac{\Delta(t-\Delta)}{t}} \left|\sup_{x \in \mathbb{R}^p} |m_{\Delta}^{(h)}(x) - m_{\Delta+1}^{(h)}(x)| - \sup_{x \in \mathbb{R}^p} |m_{\Delta}(x) - m_{\Delta+1}(x)|\right| \\
\leq&\sqrt{\frac{\Delta(t-\Delta)}{t}} \sup_{x \in \mathbb{R}^p} \left||m_{\Delta}^{(h)}(x) - m_{\Delta+1}^{(h)}(x)| - |m_{\Delta}(x) - m_{\Delta+1}(x)|\right| \\
\leq&2\sqrt{\frac{\Delta(t-\Delta)}{t}} \max\left\{\sup_{x \in \mathbb{R}^p} |m_{\Delta}^{(h)}(x) - m_{\Delta}(x)|,\ \sup_{x \in \mathbb{R}^p} |m_{\Delta+1}^{(h)}(x) - m_{\Delta+1}(x)|\right\} \\
\leq&2\sqrt{\frac{\Delta(t-\Delta)}{t}} C_{\text{Lip}} \sqrt{d} h,
\end{aligned}$$

where the last inequality follows Assumption 1. □

S.2.2 Sample terms

Lemma S.4. *Let $n \in \mathbb{N}, p \in (0,1)$ and $x \in (0,1]$. Let $(B_1, \ldots, B_n)$ be an independent sequence of* $\mathrm{Ber}(p)$ *random variables, and let $(\epsilon_1, \ldots, \epsilon_n)$ be a sequence of real-valued random variables such that, conditionally on $(B_1, \ldots, B_n)$, it holds* *(i)* $\epsilon_1, \ldots, \epsilon_n$ *are independent; and*

(ii) $\mathbb{E}(e^{\lambda \epsilon_i}) \leq e^{\lambda^2/2}$ *for all* $\lambda \in \mathbb{R}$.

Then we have

$$\mathbb{P}\left(\left|\sum_{i=1}^n \epsilon_i B_i\right| \geq x \sum_{i=1}^n B_i\right) \leq 2e^{-npx^2/4}.$$

Proof. Writing $N = \sum_{i=1}^n B_i$, we may condition on $(B_1, \ldots, B_n)$ to see that

$$\begin{aligned}\mathbb{P}\left(\frac{1}{N}\left|\sum_{i=1}^n \epsilon_i B_i\right| \geq x\right) &\leq 2\mathbb{E}\left[\mathbb{1}_{\{N \geq 1\}} e^{-\frac{Nx^2}{2}}\right] \\ &= 2\left\{\left(1 - p + pe^{-\frac{x^2}{2}}\right)^n - (1-p)^n\right\} \\ &= 2\left\{\left(1 - p(1 - e^{-\frac{x^2}{2}})\right)^n - (1-p)^n\right\}.\end{aligned}$$

For $x \in (0, 1]$ we have $1 - e^{-x^2/2} \geq x^2/4$, so that

$$\mathbb{P}\left(\frac{1}{N}\left|\sum_{i=1}^n \epsilon_i B_i\right| \geq x\right) \leq 2\left\{\left(1 - \frac{px^2}{4}\right)^n - (1-p)^n\right\} \leq 2e^{-npx^2/4}.$$

□

Lemma S.5. *Under Assumption 1, for any integer pair* (s, t), $1 \leq s < t$, *we let*

$$W_{s,t} = \sqrt{\frac{s(t-s)}{t}} \left|\max_{i=1}^t |\widetilde{m}_{1:s}(X_i) - \widetilde{m}_{(s+1):t}(X_i)| - \max_{i=1}^t |m^{(h)}_{1:s}(X_i) - m^{(h)}_{(s+1):t}(X_i)|\right|.$$

Define the following three scenarios:

(i) There exists an integer pair (s, t), $1 \leq s < t$, *such that*

$$W_{s,t} > 2\sqrt{\frac{s(t-s)}{t}} C_{\mathrm{Lip}} \sqrt{d} h + \frac{4\sigma}{\sqrt{c_{\min} h^d}} \sqrt{5\log(t) + \log(16/\gamma)}. \tag{S.9}$$

(ii) There exists an integer pair (s, t), $1 \leq s < t \leq \Delta$, *such that* (S.9) *holds.*

(iii) Under Assumption 3, there exists an integer t, $t > \Delta$, *such that*

$$W_{\Delta,t} > 2\sqrt{\frac{\Delta(t-\Delta)}{t}} C_{\mathrm{Lip}} \sqrt{d} h + \frac{4\sigma}{\sqrt{c_{\min} h^d}} \sqrt{5\log(t) + \log(16/\gamma)}.$$

We have that

- *under Assumption 2, (i) holds with probability at most* γ*; and*
- *under Assumption 3, (ii) and (iii) hold with probability at most* γ.

Proof. For any integer pairs $1 \leq s < t$ and any $\eta, \eta', \eta'' > 0$, $\eta' + \eta'' = \eta$, it holds that

$$\begin{aligned}&\mathbb{P}\left(\sqrt{\frac{s(t-s)}{t}} \left|\max_{i=1}^t |\widetilde{m}_{1:s}(X_i) - \widetilde{m}_{(s+1):t}(X_i)| - \max_{i=1}^t |m^{(h)}_{1:s}(X_i) - m^{(h)}_{(s+1):t}(X_i)|\right| \geq \eta\right) \\ &\leq \mathbb{P}\left(\sqrt{\frac{s(t-s)}{t}} \max_{i=1}^t \left||\widetilde{m}_{1:s}(X_i) - \widetilde{m}_{(s+1):t}(X_i)| - |m^{(h)}_{1:s}(X_i) - m^{(h)}_{(s+1):t}(X_i)|\right| \geq \eta\right) \\ &\leq \mathbb{P}\left(\sqrt{\frac{s(t-s)}{t}} \left\{\max_{i=1}^t |\widetilde{m}_{1:s}(X_i) - m^{(h)}_{1:s}(X_i)| + \max_{i=1}^t |\widetilde{m}_{(s+1):t}(X_i) - m^{(h)}_{(s+1):t}(X_i)|\right\} \geq \eta\right)\end{aligned}$$

$$
\begin{aligned}
&\leq \mathbb{P}\bigg(\sqrt{\frac{s(t-s)}{t}}\max_{i=1}^{t}|\widetilde{m}_{1:s}(X_i)-m_{1:s}^{(h)}(X_i)|\geq \eta'\bigg)\\
&\qquad\qquad + \mathbb{P}\bigg(\sqrt{\frac{s(t-s)}{t}}\max_{i=1}^{t}|\widetilde{m}_{(s+1):t}(X_i)-m_{(s+1):t}^{(h)}(X_i)|\geq \eta''\bigg)\\
&= (I)+(II).
\end{aligned}
\tag{S.10}
$$

Case 1. In this case, we consider $t \leq \Delta$. We first focus on term (I).

When $x \in A_{h,j}$ with a certain $j \in \mathbb{N}$, we have

$$
m_{1:s}^{(h)}(x) = \frac{\sum_{i=1}^{s}\mathbb{E}(Y_i \mathbb{1}_{\{X_i\in A_{h,j}\}})}{s\mu(A_{h,j})} = \frac{\int_{A_{h,j}} m_{\Delta}(x)\mu(dx)}{\mu(A_{h,j})}.
$$

Then, with the convention that $0/0 = 0$, we have

$$
\begin{aligned}
&\max_{i=1}^{t}|\widetilde{m}_{1:s}(X_i)-m_{1:s}^{(h)}(X_i)| = \max_{j}\bigg|\frac{\nu_{1:s}(A_{h,j})}{\mu_{1:s}(A_{h,j})} - \frac{\int_{A_{h,j}} m_{\Delta}(x)\mu(dx)}{\mu(A_{h,j})}\bigg|\\
&= \max_{j}\bigg|\frac{1}{\mu_{1:s}(A_{h,j})}\sum_{i=1}^{s}\{Y_i - m_i(X_i)\}\mathbb{1}_{\{X_i\in A_{h,j}\}}\\
&\qquad + \frac{1}{\mu_{1:s}(A_{h,j})}\sum_{i=1}^{s}\bigg(m_i(X_i) - \frac{\int_{A_{h,j}} m_{\Delta}(x)\mu(dx)}{\mu(A_{h,j})}\bigg)\mathbb{1}_{\{X_i\in A_{h,j}\}}\bigg|\\
&\leq \max_{j}\frac{1}{\mu_{1:s}(A_{h,j})}\bigg|\sum_{i=1}^{s}\{Y_i - m_i(X_i)\}\mathbb{1}_{\{X_i\in A_{h,j}\}}\bigg|\\
&\qquad + \max_{j}\frac{1}{\mu_{1:s}(A_{h,j})}\bigg|\sum_{i=1}^{s}\bigg(m_i(X_i) - \frac{\int_{A_{h,j}} m_{\Delta}(x)\mu(dx)}{\mu(A_{h,j})}\bigg)\mathbb{1}_{\{X_i\in A_{h,j}\}}\bigg|\\
&=(I.1)+(I.2).
\end{aligned}
\tag{S.11}
$$

As for the term $(I.1)$, we apply Lemma S.4 and take $B_i = \mathbb{1}_{\{X_i\in A_{h,j}\}}$. For any $\eta_1 \in (0,\sigma]$ we have

$$
\mathbb{P}\bigg(\frac{1}{\mu_{1:s}(A_{h,j})}\bigg|\sum_{i=1}^{s}\{Y_i - m_i(X_i)\}\mathbb{1}_{\{X_i\in A_{h,j}\}}\bigg|\geq \eta_1\bigg) \leq 2\exp\bigg(-\frac{s\mu(A_{h,j})\eta_1^2}{4\sigma^2}\bigg).
\tag{S.12}
$$

As for the term $(I.2)$, when $x \in A_{h,j}$, due to Assumption 1, we have

$$
\bigg|m_i(x) - \frac{\int_{A_{h,j}} m_i(x')\mu(dx')}{\mu(A_{h,j})}\bigg| \leq C_{\mathrm{Lip}}\mathrm{diam}(A_{h,j}) = C_{\mathrm{Lip}}\sqrt{d}h,
$$

hence with probability one, it holds that

$$
\begin{aligned}
&\frac{1}{\mu_{1:s}(A_{h,j})}\bigg|\sum_{i=1}^{s}\bigg(m_i(X_i) - \frac{\int_{A_{h,j}} m_{\Delta}(x)\mu(dx)}{\mu(A_{h,j})}\bigg)\mathbb{1}_{\{X_i\in A_{h,j}\}}\bigg|\\
\leq&\frac{1}{\mu_{1:s}(A_{h,j})}\sum_{i=1}^{s}\bigg|m_i(X_i) - \frac{\int_{A_{h,j}} m_{\Delta}(x)\mu(dx)}{\mu(A_{h,j})}\bigg|\mathbb{1}_{\{X_i\in A_{h,j}\}} \leq C_{\mathrm{Lip}}\sqrt{d}h.
\end{aligned}
\tag{S.13}
$$

Combining (S.11), (S.12) and (S.13), with a union bound argument, we have that, for any $\eta_1 \in (0,\sigma]$

$$
\begin{aligned}
&\mathbb{P}\bigg\{\sqrt{\frac{s(t-s)}{t}}\max_{i=1}^{t}|\widetilde{m}_{1:s}(X_i)-m_{1:s}^{(h)}(X_i)|\geq \sqrt{\frac{s(t-s)}{t}}\Big(\eta_1 + C_{\mathrm{Lip}}\sqrt{d}h\Big)\bigg\}\\
&\qquad\qquad \leq 2t\exp\bigg(-\frac{s\mu(A_{h,j})\eta_1^2}{4\sigma^2}\bigg).
\end{aligned}
\tag{S.14}
$$

For any $\eta_2 \in (0, \sigma]$, almost identical arguments lead to

$$\mathbb{P}\left\{\sqrt{\frac{s(t-s)}{t}} \max_{i=1}^{t} |\widetilde{m}_{(s+1):t}(X_i) - m^{(h)}_{(s+1):t}(X_i)| \geq \sqrt{\frac{s(t-s)}{t}} \left(\eta_2 + C_{\mathrm{Lip}}\sqrt{d}h\right)\right\}$$
$$\leq 2t \exp\left(-\frac{(t-s)\mu(A_{h,j})\eta_2^2}{4\sigma^2}\right). \tag{S.15}$$

Combining (S.10), (S.14) and (S.15), we have that

$$\begin{aligned}
&\mathbb{P}\Bigg\{\sqrt{\frac{s(t-s)}{t}} \left|\max_{i=1}^{t} |\widetilde{m}_{1:s}(X_i) - \widetilde{m}_{(s+1):t}(X_i)| - \max_{i=1}^{t} |m^{(h)}_{1:s}(X_i) - m^{(h)}_{(s+1):t}(X_i)|\right| \\
&\qquad \geq \sqrt{\frac{s(t-s)}{t}}(\eta_1 + \eta_2) + 2\sqrt{\frac{s(t-s)}{t}} C_{\mathrm{Lip}}\sqrt{d}h\Bigg\} \\
&\leq 2t \exp\left(-\frac{s\mu(A_{h,j})\eta_1^2}{4\sigma^2}\right) + 2t \exp\left(-\frac{(t-s)\mu(A_{h,j})\eta_2^2}{4\sigma^2}\right) \\
&\leq 2t \exp\left(-\frac{s c_{\min} h^d \eta_1^2}{4\sigma^2}\right) + 2t \exp\left(-\frac{(t-s) c_{\min} h^d \eta_2^2}{4\sigma^2}\right),
\end{aligned}$$

where the last inequality follows from Assumption 1.

We let

$$\begin{aligned}
Q_{s,t} = &\sqrt{\frac{s(t-s)}{t}} \left|\max_{i=1}^{t} |\widetilde{m}_{1:s}(X_i) - \widetilde{m}_{(s+1):t}(X_i)| - \max_{i=1}^{t} |m^{(h)}_{1:s}(X_i) - m^{(h)}_{(s+1):t}(X_i)|\right| \\
&- 2\sqrt{\frac{s(t-s)}{t}} C_{\mathrm{Lip}}\sqrt{d}h,
\end{aligned}$$

$$\eta_1 = \sqrt{\frac{4\sigma^2}{s c_{\min} h^d}}\varepsilon_t \quad \text{and} \quad \eta_2 = \sqrt{\frac{4\sigma^2}{(t-s) c_{\min} h^d}}\varepsilon_t,$$

with $\varepsilon_t > 0$ to be specified. Therefore

$$\mathbb{P}\left\{Q_{s,t} \geq \frac{4\sigma\varepsilon_t}{\sqrt{c_{\min} h^d}}\right\} \leq \mathbb{P}\left\{Q_{s,t} \geq \sqrt{\frac{s(t-s)}{t}}(\eta_1 + \eta_2)\right\} \leq 4t \exp(-\varepsilon_t^2).$$

Case 1.1 When $\Delta = \infty$, we have that

$$\begin{aligned}
&\mathbb{P}\left\{\exists s, t \in \mathbb{N}^*, t > 1, s \in [1, t) : \, Q_{s,t} \geq \frac{4\sigma\varepsilon_t}{\sqrt{c_{\min} h^d}}\right\} \\
\leq &\sum_{j=1}^{\infty} \mathbb{P}\left\{\max_{2^j \leq t < 2^{j+1}} \max_{1 \leq s < t} Q_{s,t} \geq \frac{4\sigma\varepsilon_t}{\sqrt{c_{\min} h^d}}\right\} \leq \sum_{j=1}^{\infty} 2^j \max_{2^j \leq t < 2^{j+1}} \mathbb{P}\left\{\max_{1 \leq s < t} Q_{s,t} \geq \frac{4\sigma\varepsilon_t}{\sqrt{c_{\min} h^d}}\right\} \\
\leq &\sum_{j=1}^{\infty} 2^j \max_{2^j \leq t < 2^{j+1}} 4t \max_{1 \leq s < t} \mathbb{P}\left\{Q_{s,t} \geq \frac{4\sigma\varepsilon_t}{\sqrt{c_{\min} h^d}}\right\} \leq 4\sum_{j=1}^{\infty} 2^j \max_{2^j \leq t < 2^{j+1}} t^2 \exp(-\varepsilon_t^2) \\
\leq &4\sum_{j=1}^{\infty} 2^{3j+2} \exp(-\varepsilon_{2^j}^2) \leq \gamma,
\end{aligned}$$

where the last inequality holds by taking

$$\varepsilon_t = \sqrt{5\log(t) + \log(16/\gamma)}. \tag{S.16}$$

Case 1.2 When $\Delta < \infty$, with ε_t defined in (S.16), it holds that

$$\mathbb{P}_{\Delta}\left\{\exists s, t \in \mathbb{N}^*, 1 \leq s < t \leq \Delta : \, Q_{s,t} \geq \frac{4\sigma\varepsilon_t}{\sqrt{c_{\min} h^d}}\right\}$$

$$\leq \mathbb{P}_\infty \left\{ \exists s, t \in \mathbb{N}^*, t > 1, s \in [1, t) : \, Q_{s,t} \geq \frac{4\sigma\varepsilon_t}{\sqrt{c_{\min} h^d}} \right\} \leq \gamma.$$

Case 2. In this case, we consider $s = \Delta < t$. Note that for any such integer pair (s, t), within both intervals $[1 : s]$ and $[(s+1) : t]$, there is one and only one underlying distribution. Therefore, based on identical arguments as those in **Case 1.**, we have that

$$\mathbb{P} \left\{ Q_{\Delta,t} \geq \frac{4\sigma\varepsilon_t}{\sqrt{c_{\min} h^d}} \right\} \leq 4t \exp(-\varepsilon_t^2).$$

Then we have

$$\mathbb{P}_\Delta \left\{ \exists t \in \mathbb{N}^*, t > \Delta : \, Q_{\Delta,t} \geq \frac{4\sigma\varepsilon_t}{\sqrt{c_{\min} h^d}} \right\} \leq \gamma.$$

We then complete the proof. □

Lemma S.6 (Laplace concentration)**.** *Let $\epsilon_1, \ldots, \epsilon_n$ be independent standard Laplace-distributed random variables (mean zero, variance 2). Then for all $x > 0$ we have*

$$\mathbb{P}\biggl(\frac{1}{n} \sum_{i=1}^n \epsilon_i \geq x \biggr) \leq \exp\biggl(-\frac{3nx^2}{4 + 3x} \biggr).$$

Note that this implies Lemma 1 of [1].

Proof. Using a Chernoff bound and taking $\lambda = \frac{n}{x}(\sqrt{1+x^2} - 1)$ we have

$$\begin{aligned}
\mathbb{P}\biggl(\frac{1}{n} \sum_{i=1}^n \epsilon_i \geq x \biggr) &\leq \inf_{\lambda \in (0,n)} e^{-\lambda x} \biggl(1 - \frac{\lambda^2}{n^2} \biggr)^{-n} \\
&\leq \exp\biggl(-n(\sqrt{1+x^2} - 1) - n \log\Bigl(1 - \frac{2 + x^2 - 2\sqrt{1+x^2}}{x^2} \Bigr) \biggr) \\
&= \exp\biggl(-n \biggl\{ \frac{x^2}{1 + \sqrt{1+x^2}} - \log\biggl(\frac{2}{1 + \sqrt{1+x^2}} \biggr) \biggr\} \biggr).
\end{aligned}$$

It is a fact (checked numerically) that

$$\frac{x^2}{1 + \sqrt{1+x^2}} - \log\biggl(\frac{2}{1 + \sqrt{1+x^2}} \biggr) \geq \frac{3x^2}{4 + 3x}$$

for all $x \geq 0$, and the result follows. □

Lemma S.7 (Private version of Lemma S.5)**.** *Suppose that*

$$\frac{s(t-s)}{t} c_{\min}^2 h^{2d} \alpha^2 \geq 64 \log\biggl(\frac{72t^3}{\gamma c_{\min} h^d} \biggr).$$

Then, if either $1 \leq s < t \leq \Delta$ or $\Delta = s < t$, with probability at least $1 - \gamma/(2t^3)$ we have

$$\begin{aligned}
\sqrt{\frac{s(t-s)}{t}} \biggl| \sup_{x \in \mathbb{R}^d} |\widehat{m}_{1:s}(x) - \widehat{m}_{(s+1):t}(x)| - \sup_{x \in \mathbb{R}^d} |m_{1:s}^{(h)}(x) - m_{(s+1):t}^{(h)}(x)| \biggr| \\
\leq 2\sqrt{\frac{s(t-s)}{t}} C_{\mathrm{Lip}} \sqrt{d} h + \frac{2^9 M}{c_{\min} h^d \alpha} \sqrt{\log\biggl(\frac{72t^3}{\gamma c_{\min} h^d} \biggr)}.
\end{aligned}$$

Proof. We start by writing

$$\biggl| \sup_{x \in \mathbb{R}^d} |\widehat{m}_{1:s}(x) - \widehat{m}_{(s+1):t}(x)| - \sup_{x \in \mathbb{R}^d} |m_{1:s}^{(h)}(x) - m_{(s+1):t}^{(h)}(x)| \biggr|$$

$$\leq \sup_{x\in\mathbb{R}^d} |\widehat{m}_{1:s}(x) - m^{(h)}_{1:s}(x)| + \sup_{x\in\mathbb{R}^d} |\widehat{m}_{(s+1):t}(x) - m^{(h)}_{(s+1):t}(x)|$$

As in the proof of Lemma S.5, it suffices to consider the first term in the case that $s \leq \Delta$. This is given by

$$\max_j \biggl| \frac{\widehat{\nu}_{1:s}(A_{h,j})}{\widehat{\mu}_{1:s}(A_{h,j})} \mathbb{1}_{\{\widehat{\mu}_{1:s}(A_{h,j}) \geq \log(n)/n\}} - \frac{\int_{A_{h,j}} m_\Delta(x)\mu(dx)}{\mu(A_{h,j})} \biggr|.$$

For any j with $\widehat{\mu}_{1:s}(A_{h,j}) \geq \log(n)/n$ we have

$$\begin{aligned}
&\biggl| \frac{\widehat{\nu}_{1:s}(A_{h,j})}{\widehat{\mu}_{1:s}(A_{h,j})} - \frac{\int_{A_{h,j}} m_\Delta(x)\mu(dx)}{\mu(A_{h,j})} \biggr| \\
&\leq |\widehat{\nu}_{1:s}(A_{h,j})| \biggl| \frac{1}{\widehat{\mu}_{1:s}(A_{h,j})} - \frac{1}{\mu_{1:s}(A_{h,j})} \biggr| + \frac{|\widehat{\nu}_{1:s}(A_{h,j}) - \nu_{1:s}(A_{h,j})|}{\mu_{1:s}(A_{h,j})} + \biggl| \frac{\nu_{1:s}(A_{h,j})}{\mu_{1:s}(A_{h,j})} - \frac{\int_{A_{h,j}} m_\Delta(x)\mu(dx)}{\mu(A_{h,j})} \biggr| \\
&= \biggl| \frac{\widehat{\nu}_{1:s}(A_{h,j})}{\widehat{\mu}_{1:s}(A_{h,j})} \biggr| \biggl| \frac{(4/\alpha)\sum_{i=1}^s \epsilon_{i,j}}{\sum_{i=1}^s \mathbb{1}_{\{X_i\in A_{h,j}\}}} \biggr| + \biggl| \frac{(4M/\alpha)\sum_{i=1}^s \zeta_{i,j}}{\sum_{i=1}^s \mathbb{1}_{\{X_i\in A_{h,j}\}}} \biggr| + \biggl| \frac{\nu_{1:s}(A_{h,j})}{\mu_{1:s}(A_{h,j})} - \frac{\int_{A_{h,j}} m_\Delta(x)\mu(dx)}{\mu(A_{h,j})} \biggr|.
\end{aligned}$$

The final term can be bounded exactly as in the proof of Lemma S.5. For the first two terms

$$\begin{aligned}
&\mathbb{P}\biggl(\biggl| \frac{\widehat{\nu}_{1:s}(A_{h,j})}{\widehat{\mu}_{1:s}(A_{h,j})} \biggr| \biggl| \frac{(4/\alpha)\sum_{i=1}^s \epsilon_{i,j}}{\sum_{i=1}^s \mathbb{1}_{\{X_i\in A_{h,j}\}}} \biggr| + \biggl| \frac{(4M/\alpha)\sum_{i=1}^s \zeta_{i,j}}{\sum_{i=1}^s \mathbb{1}_{\{X_i\in A_{h,j}\}}} \biggr| \geq \eta \biggr) \\
&\leq \mathbb{P}\biggl(\biggl| \frac{\widehat{\nu}_{1:s}(A_{h,j})}{\widehat{\mu}_{1:s}(A_{h,j})} \biggr| \biggl| \frac{(4/\alpha)\sum_{i=1}^s \epsilon_{i,j}}{\sum_{i=1}^s \mathbb{1}_{\{X_i\in A_{h,j}\}}} \biggr| \geq \eta/2 \biggr) + \mathbb{P}\biggl(\biggl| \frac{(4M/\alpha)\sum_{i=1}^s \zeta_{i,j}}{\sum_{i=1}^s \mathbb{1}_{\{X_i\in A_{h,j}\}}} \biggr| \geq \eta/2 \biggr) \\
&\leq \mathbb{P}\biggl(\biggl| \frac{\widehat{\nu}_{1:s}(A_{h,j})}{\widehat{\mu}_{1:s}(A_{h,j})} \biggr| \geq 4M \biggr) + \mathbb{P}\biggl(\biggl| \frac{(4/\alpha)\sum_{i=1}^s \epsilon_{i,j}}{\sum_{i=1}^s \mathbb{1}_{\{X_i\in A_{h,j}\}}} \biggr| \geq \eta/(8M) \biggr) + \mathbb{P}\biggl(\biggl| \frac{(4M/\alpha)\sum_{i=1}^s \zeta_{i,j}}{\sum_{i=1}^s \mathbb{1}_{\{X_i\in A_{h,j}\}}} \biggr| \geq \eta/2 \biggr) \\
&\leq \mathbb{P}\biggl(\biggl| \frac{\sum_{i=1}^s Y_i \mathbb{1}_{\{X_i\in A_{h,j}\}} + \frac{4M}{\alpha}\sum_{i=1}^s \zeta_{i,j}}{\sum_{i=1}^s \mathbb{1}_{\{X_i\in A_{h,j}\}} + \frac{4}{\alpha}\sum_{i=1}^s \epsilon_{i,j}} \biggr| \geq 4M \biggr) + 2\mathbb{P}\biggl(\biggl| \frac{\sum_{i=1}^s \epsilon_{i,j}}{\sum_{i=1}^s \mathbb{1}_{\{X_i\in A_{h,j}\}}} \biggr| \geq \eta\alpha/(32M) \biggr) \\
&\leq \mathbb{P}\biggl(\frac{4}{\alpha}\sum_{i=1}^s \epsilon_{i,j} < -\frac{1}{2}\sum_{i=1}^s \mathbb{1}_{\{X_i\in A_{h,j}\}} \biggr) + \mathbb{P}\biggl(\frac{|\sum_{i=1}^s Y_i \mathbb{1}_{\{X_i\in A_{h,j}\}} + \frac{4M}{\alpha}\sum_{i=1}^s \zeta_{i,j}|}{\sum_{i=1}^s \mathbb{1}_{\{X_i\in A_{h,j}\}}} \geq 2M \biggr) \\
&\qquad\qquad + 2\mathbb{P}\biggl(\biggl| \frac{\sum_{i=1}^s \epsilon_{i,j}}{\sum_{i=1}^s \mathbb{1}_{\{X_i\in A_{h,j}\}}} \biggr| \geq \eta\alpha/(32M) \biggr) \\
&\leq 4\mathbb{P}\biggl(\biggl| \sum_{i=1}^s \epsilon_{i,j} \biggr| \geq \alpha \min\bigl\{\eta/(32M), 1/8\bigr\} \sum_{i=1}^s \mathbb{1}_{\{X_i\in A_{h,j}\}} \biggr) \\
&\leq 4\mathbb{P}\biggl(\biggl| \frac{1}{s}\sum_{i=1}^s \epsilon_{i,j} \biggr| \geq \alpha\mu(A_{h,j}) \min\bigl\{\eta/(64M), 1/16\bigr\} \biggr) + 4\mathbb{P}\biggl(\sum_{i=1}^s \mathbb{1}_{\{X_i\in A_{h,j}\}} < s\mu(A_{h,j})/2 \biggr).
\end{aligned}$$

We have thus reduced our problem to two standard concentration inequalities. By Lemma S.6 we have

$$\mathbb{P}\biggl(\biggl| \sum_{i=1}^s \epsilon_{i,j} \biggr| \geq \alpha\mu(A_{h,j}) \min\bigl\{\eta/(64M), 1/16\bigr\} \biggr) \leq 2\exp\Bigl(-s\alpha^2\mu(A_{h,j})^2 \min\{\eta^2/(2^{14}M^2), 2^{-6}\} \Bigr).$$

Using the fact that $e^{-x} \leq 1 - x + x^2/2$ for all $x > 0$ we have the Chernoff bound

$$\begin{aligned}
\mathbb{P}\biggl(\sum_{i=1}^s \mathbb{1}_{\{X_i\in A_{h,j}\}} < s\mu(A_{h,j})/2 \biggr) &\leq \inf_{\lambda>0} e^{\lambda\mu(A_{h,j})/2} \{1 - \mu(A_{h,j}) + \mu(A_{h,j})e^{-\lambda/s}\}^s \\
&\leq \inf_{\lambda>0} e^{\lambda\mu(A_{h,j})/2} \{1 - \lambda\mu(A_{h,j})/s + \lambda^2\mu(A_{h,j})/(2s^2)\}^s \\
&\leq \inf_{\lambda>0} \exp\Bigl(-\lambda\mu(A_{h,j})/2 + \lambda^2\mu(A_{h,j})/(2s) \Bigr) \\
&= \exp\Bigl(-\frac{s\mu(A_{h,j})}{8} \Bigr).
\end{aligned}$$

Hence,

$$\mathbb{P}\biggl(\Bigl|\frac{\widehat{\nu}_{1:s}(A_{h,j})}{\widehat{\mu}_{1:s}(A_{h,j})}\Bigr|\Bigl|\frac{(4/\alpha)\sum_{i=1}^{s}\epsilon_{i,j}}{\sum_{i=1}^{s}\mathbb{1}_{\{X_i\in A_{h,j}\}}}\Bigr| + \Bigl|\frac{(4M/\alpha)\sum_{i=1}^{s}\zeta_{i,j}}{\sum_{i=1}^{s}\mathbb{1}_{\{X_i\in A_{h,j}\}}}\Bigr| \geq \epsilon\biggr)$$
$$\leq 16\exp\biggl(-s\alpha^2\mu(A_{h,j})^2\min\{\epsilon^2/(2^{14}M^2), 2^{-6}\}\biggr).$$

Combining the previous bound with bounds from the proof of Lemma S.5, when $\eta \in (0, 2^6M\sqrt{s(t-s)/t}]$ we have

$$\mathbb{P}\biggl(\sqrt{\frac{s(t-s)}{t}}\Bigl|\sup_{x\in\mathbb{R}^d}|\widehat{m}_{1:s}(x)-\widehat{m}_{(s+1):t}(x)| - \sup_{x\in\mathbb{R}^d}|m^{(h)}_{1:s}(x)-m^{(h)}_{(s+1):t}(x)|\Bigr| \geq \eta + 2\sqrt{\frac{s(t-s)}{t}}C_{\mathrm{Lip}}\sqrt{d}h\biggr)$$
$$\leq \mathbb{P}\biggl(\sqrt{\frac{s(t-s)}{t}}\sup_x\bigl|\widehat{m}_{1:s}(x)-m^{(h)}_{1:s}(x)\bigr| \geq \eta/4 + \sqrt{\frac{s(t-s)}{t}}C_{\mathrm{Lip}}\sqrt{d}h\biggr)\biggr)$$
$$+ \mathbb{P}\biggl(\sqrt{\frac{s(t-s)}{t}}\sup_x\bigl|\widehat{m}_{(s+1):t}(x)-m^{(h)}_{(s+1):t}(x)\bigr| \geq \eta/4 + \sqrt{\frac{s(t-s)}{t}}C_{\mathrm{Lip}}\sqrt{d}h\biggr)\biggr)$$
$$+ \mathbb{P}\biggl(\sqrt{\frac{s(t-s)}{t}}\max_j\biggl\{\Bigl|\frac{\widehat{\nu}_{1:s}(A_{h,j})}{\widehat{\mu}_{1:s}(A_{h,j})}\Bigr|\Bigl|\frac{(4/\alpha)\sum_{i=1}^{s}\epsilon_{i,j}}{\sum_{i=1}^{s}\mathbb{1}_{\{X_i\in A_{h,j}\}}}\Bigr| + \Bigl|\frac{(4M/\alpha)\sum_{i=1}^{s}\zeta_{i,j}}{\sum_{i=1}^{s}\mathbb{1}_{\{X_i\in A_{h,j}\}}}\Bigr|\biggr\} \geq \eta/4\biggr)$$
$$+ \mathbb{P}\biggl(\sqrt{\frac{s(t-s)}{t}}\max_j\biggl\{\Bigl|\frac{\widehat{\nu}_{(s+1):t}(A_{h,j})}{\widehat{\mu}_{(s+1):t}(A_{h,j})}\Bigr|\Bigl|\frac{(4/\alpha)\sum_{i=s+1}^{t}\epsilon_{i,j}}{\sum_{i=s+1}^{t}\mathbb{1}_{\{X_i\in A_{h,j}\}}}\Bigr| + \Bigl|\frac{(4M/\alpha)\sum_{i=s+1}^{t}\zeta_{i,j}}{\sum_{i=s+1}^{t}\mathbb{1}_{\{X_i\in A_{h,j}\}}}\Bigr|\biggr\} \geq \eta/4\biggr)$$
$$\leq \frac{2}{c_{\min}h^d}\biggl\{\exp\biggl(-\frac{t}{t-s}\frac{c_{\min}h^d\eta^2}{64M^2}\biggr) + \exp\biggl(-\frac{t}{s}\frac{c_{\min}h^d\eta^2}{64M^2}\biggr) + 8\exp\biggl(-s\alpha^2c_{\min}^2h^{2d}\min\Bigl\{\frac{t\eta^2}{s(t-s)2^{18}M^2}, 2^{-6}\Bigr\}\biggr)$$
$$+ 8\exp\biggl(-(t-s)\alpha^2c_{\min}^2h^{2d}\min\Bigl\{\frac{t\eta^2}{s(t-s)2^{18}M^2}, 2^{-6}\Bigr\}\biggr)\biggr\}$$
$$\leq \frac{4}{c_{\min}h^d}\biggl\{\exp\biggl(-\frac{c_{\min}h^d\eta^2}{64M^2}\biggr) + 8\exp\biggl(-\frac{\alpha^2c_{\min}^2h^{2d}\eta^2}{2^{18}M^2}\biggr)\biggr\} \leq \frac{36}{c_{\min}h^d}\exp\biggl(-\frac{\alpha^2c_{\min}^2h^{2d}\eta^2}{2^{18}M^2}\biggr).$$

The result follows on taking

$$\eta = \frac{2^9M}{c_{\min}h^d\alpha}\sqrt{\log(72t^3)+\log(1/c_{\min})+\log(h^{-d})+\log(1/\gamma)}.$$

□

S.3 Proofs of the results in Appendix A

Lemma S.8. *For any $\gamma > 0$, it holds that*

$$\mathbb{P}\biggl\{\exists s,t\in\mathbb{N},\ t>1,\ s\in[1,t):\ \biggl|\Bigl(\frac{t-s}{ts}\Bigr)^{1/2}\sum_{l=1}^{s}(Z_l-f_l) - \Bigl\{\frac{s}{t(t-s)}\Bigr\}^{1/2}\sum_{l=s+1}^{t}(Z_l-f_l)\biggr|$$
$$> 2^{3/2}\sqrt{\sigma^2+4\alpha^{-2}}\log^{1/2}(t/\gamma)\biggr\} \leq \gamma.$$

Proof. It holds that for any sequence $\{\varepsilon_t > 0\}$,

$$\mathbb{P}\biggl\{\exists s,t\in N,\ t>1,\ s\in[1,t):\ \biggl|\Bigl(\frac{t-s}{ts}\Bigr)^{1/2}\sum_{l=1}^{s}(Z_l-f_l) - \Bigl\{\frac{s}{t(t-s)}\Bigr\}^{1/2}\sum_{l=s+1}^{t}(Z_l-f_l)\biggr| \geq \varepsilon_t\biggr\}$$
$$\leq \sum_{j=1}^{\infty}\mathbb{P}\biggl\{\max_{2^j\leq t<2^{j+1}}\max_{1\leq s<t}\biggl|\Bigl(\frac{t-s}{ts}\Bigr)^{1/2}\sum_{l=1}^{s}(Z_l-f_l) - \Bigl\{\frac{s}{t(t-s)}\Bigr\}^{1/2}\sum_{l=s+1}^{t}(Z_l-f_l)\biggr| \geq \varepsilon_t\biggr\}$$

$$\leq \sum_{j=1}^{\infty} 2^j \max_{2^j \leq t < 2^{j+1}} \mathbb{P}\left\{ \max_{1\leq s<t} \left| \left(\frac{t-s}{ts}\right)^{1/2} \sum_{l=1}^{s}(Z_l - f_l) - \left\{\frac{s}{t(t-s)}\right\}^{1/2} \sum_{l=s+1}^{t}(Z_l - f_l)\right| \geq \varepsilon_t \right\}$$

$$\leq \sum_{j=1}^{\infty} 2^j \max_{2^j \leq t < 2^{j+1}} t\mathbb{P}\left\{|W| \geq \varepsilon_t\right\} \leq \sum_{j=1}^{\infty} 2^{2j+1}\mathbb{P}\left\{|W| \geq \varepsilon'_j\right\},$$

where $\varepsilon_t = \varepsilon'_j$, for any $t \in \{2^j, \ldots, 2^{j+1}-1\}$, $j = 1, 2, \ldots$ and W is a mean zero sub-Gaussian random variable with

$$\|W\|_{\psi_2} \leq \sqrt{\sigma^2 + 4\alpha^{-2}}. \tag{S.17}$$

We remark that (S.17) is due to Assumption 6, which implies that $\mathbb{E}(Z_l) = f_l$ and

$$\|Z_l - f_l\|_{\psi_2} \leq \sqrt{\|X_l - f_l\|_{\psi_2}^2 + \|\epsilon_l\|_{\psi_2}^2} = \sqrt{\sigma^2 + 4\alpha^{-2}},$$

where the last identity follows from Lemma 1 in [1].

For any $t = 1, 2, \ldots$, let

$$\varepsilon_t = \sqrt{2\sigma^2 + 8\alpha^{-2}}\left[2\log(t) + \mathrm{loglog}(t) + \log\left\{\log(t) + \log(2)\right\} - 2\,\mathrm{loglog}(2) - \log(\gamma)\right]^{1/2}.$$

Due to the sub-Gaussianity, we have that for any $\zeta > 0$, $\mathbb{P}\left\{|W| \geq \zeta\right\} < 2\exp(-2^{-1}\zeta^2/(\sigma^2+4\alpha^{-2}))$ [e.g. (2.9) in 3].

$$\mathbb{P}\left\{\exists s, t \in N,\, t > 1,\, s \in [1, t) : \left| \left(\frac{t-s}{ts}\right)^{1/2} \sum_{l=1}^{s}(Z_l - f_l) - \left\{\frac{s}{t(t-s)}\right\}^{1/2} \sum_{l=s+1}^{t}(Z_l - f_l)\right| \geq \varepsilon_t \right\}$$

$$\leq \sum_{j=1}^{\infty} \max_{2^j \leq t \leq 2^{j+1}} \exp[(2j+2)\log(2) - 2\log(t) - \mathrm{loglog}(t) - \log\{\log(t) + \log(2)\} + 2\,\mathrm{loglog}(2) + \log(\gamma)]$$

$$\leq \sum_{j=1}^{\infty} \exp\left\{(2j+2)\log(2) - 2j\log(2) - 2\log(j) - \log\{(j+1)\log(2)\} + 2\,\mathrm{loglog}(2) + \log(\gamma)\right\}$$

$$\leq \gamma \sum_{j=1}^{\infty} \frac{1}{j(j+1)} \leq \gamma.$$

For simplicity, we let

$$\varepsilon_t = 2^{3/2}\sqrt{\sigma^2 + 4\alpha^{-2}}\log^{1/2}(t/\gamma),$$

which satisfies that for any $t \geq 2$ and $\gamma \in (0, 1)$,

$$2^{3/2}\log^{1/2}(t/\gamma) \geq \sqrt{2}[2\log(t) + \mathrm{loglog}(t) + \log\left\{\log(t) + \log(2)\right\} - 2\,\mathrm{loglog}(2) - \log(\gamma)]^{1/2}. \tag{S.18}$$

We therefore completes the proof. □

Proof of Theorem 5. **Step 1.** Define the event

$$\mathcal{B} = \left\{\forall s, t \in N,\, t > 1,\, s \in [1, t) : \left| \left(\frac{t-s}{ts}\right)^{1/2} \sum_{l=1}^{s}(Z_l - f_l) - \left\{\frac{s}{t(t-s)}\right\}^{1/2} \sum_{l=s+1}^{t}(Z_l - f_l)\right| < b_t \right\}. \tag{S.19}$$

It follows from Lemma S.8 that $\mathbb{P}\{\mathcal{B}\} > 1 - \gamma$. Throughout the proof we assume that the event $\mathcal{B}$ holds.

For any $s, t \in \mathbb{N}$, $1 \leq s < t$, it holds that $\left|\widehat{D}_{s,t} - D_{s,t}\right| < b_t$, which implies that

$$D_{s,t} + b_t > \widehat{D}_{s,t} > D_{s,t} - b_t. \tag{S.20}$$

Step 2. For any $t \leq \Delta$, we have that $D_{s,t} = 0$, for all $s \in [1, t)$. Thus, using (S.20), we conclude that, $\widehat{t} > t$ and, therefore that $\widehat{t} > \Delta$.

Step 3. Now we consider any $t > \Delta$. If there exists $s \in [1, t)$ such that $\widehat{D}_{s,t} > b_t$, then $d \leq t - \Delta$. Thus, $d \leq \widetilde{t} - \Delta$, where

$$\widetilde{t} = \min\{t > \Delta, \exists s \in [\Delta, t), \widehat{D}_{s,t} > b_t\},$$

and any upper bound on $\widetilde{t}$ will also be an upper bound on d, when the signal-to-noise constraint specified by Assumption 8 is satisfied. Thus, our task becomes that of computing a sharp upper bound on $\widetilde{t}$. To that effect, notice that, when $\Delta \leq s < t$,

$$D_{s,t} = \Delta \left(\frac{t-s}{ts}\right)^{1/2} |\mu_1 - \mu_2| = \Delta \left(\frac{t-s}{ts}\right)^{1/2} \kappa, \tag{S.21}$$

and, because of (S.20) again,

$$\widehat{D}_{s,t} \geq \Delta \left(\frac{t-s}{ts}\right)^{1/2} \kappa - b_t.$$

As a result, we obtain that $\widetilde{t} \leq t^*$, where

$$t_* = \min\left\{t > \Delta : \max_{s \in [\Delta, t)} \left\{\Delta \left(\frac{t-s}{ts}\right)^{1/2} \kappa - 2b_t\right\} \geq 0\right\}.$$

Step 4. Write for convenience $m = t^* - \Delta$, so that $\widehat{t} - \Delta \leq m$. Recalling that

$$b_t = 2^{3/2}\sqrt{\sigma^2 + 4\alpha^{-2}} \log^{1/2}(t/\gamma),$$

we seek the smallest integer m such that

$$\max_{s \in [\Delta, m+\Delta)} \left[\Delta\kappa \left\{\frac{m + \Delta - s}{(m+\Delta)s}\right\}^{1/2} - 2^{5/2}\sqrt{\sigma^2 + 4\alpha^{-2}} \log^{1/2}\{(m+\Delta)/\gamma\}\right] > 0,$$

which is equivalent to finding the smallest integer m such that

$$\max_{s \in [\Delta, m+\Delta)} \left[\Delta^2\kappa^2 - 32(\sigma^2 + 4\alpha^{-2})\frac{s(m+\Delta)}{m+\Delta-s} \log\{(m+\Delta)/\gamma\}\right] > 0.$$

In turn, the above task corresponds to that of computing the smallest integer m such that

$$\begin{aligned}\Delta^2 \frac{\kappa^2}{\sigma^2 + 4\alpha^{-2}} &> \min_{s \in [\Delta, m+\Delta)} \left[32\frac{s(m+\Delta)}{m+\Delta-s} \log\{(m+\Delta)/\gamma\}\right] \\ &= 32\frac{\Delta(m+\Delta)}{m} \log\{(m+\Delta)/\gamma\},\end{aligned}$$

or, equivalently, such that

$$m\left[\frac{\Delta\kappa^2}{32(\sigma^2 + 4\alpha^{-2})} - \log\{(m+\Delta)/\gamma\}\right] > \Delta \log\{(m+\Delta)/\gamma\}, \tag{S.22}$$

under Assumption 8.

Let C_d be an absolute constant large enough and also upper bounded by C_{SNR}. The claimed result now follows once we show that that the value

$$m^* = \lceil C_d \log(\Delta/\gamma)(\sigma^2 + 4\alpha^{-2})\kappa^{-2}\rceil$$

satisfies (S.22). To see this, assume for simplicity that $C_d \log(\Delta/\gamma)(\sigma^2 + 4\alpha^{-2})\kappa^{-2}$ is an integer; if not, the proof only requires trivial modifications. We first point out that $m^* \leq \Delta$ because of Assumption 8 and the fact that $C_d \leq C_{\mathrm{SNR}}$. Now, the left hand side of inequality (S.22) is equal, for this choice of m, to

$$C_d \log(\Delta/\gamma)\frac{\Delta}{32} - C_d\frac{\sigma^2 + 4\alpha^{-2}}{\kappa^2} \log(\Delta/\gamma) \log\left\{\frac{C_d \log(\Delta/\gamma)(\sigma^2 + 4\alpha^{-2})/\kappa^2 + \Delta}{\gamma}\right\}. \tag{S.23}$$

Using again Assumption 8 and the fact that $m^* \leq \Delta$, the second term in the previous expression is upper bounded by

$$\frac{2C_d}{C_{\mathrm{SNR}}}\Delta \log(\Delta/\gamma),$$

due to the fact that $2\log(x) \geq \log(2x)$, $x \geq 2$. Thus, the quantity in (S.23) is lower bounded by

$$\Delta \log(\Delta/\gamma) \left(\frac{C_d}{32} - \frac{2C_d}{C_{\mathrm{SNR}}} \right) \geq 2\Delta \log(\Delta/\gamma) \geq \Delta \log(2\Delta/\gamma) \geq \Delta \log\left((m^* + \Delta)/\gamma\right),$$

where the first inequality is justified by first choosing a large enough C_d and then choosing C_{SNR} larger than C_d, and the second and third inequalities follow from $\log(\Delta/\gamma) \geq 0$ and $m^* \leq \Delta$, respectively. Thus, combining the last display with (S.22) and (S.23) yields (16). Finally, (14) and (15) follow immediately from Steps 1 and 2. □

Proof of Theorem 6. Throughout the proof we will assume for simplicity that $c\sigma^2\alpha^{-2}\log(1/\gamma)\kappa^{-2}$ is an integer.

Step 1. For any n, let P^n be the restrictions of a distribution P to $\mathcal{F}_n$, i.e. the σ-field generated by the observations $\{Z_i\}_{i=1}^n$.

Let P_1 be the distribution of the original X_t before the change and let P_2 be the distribution after. Write

$$m_{t,k}(z|z_1, \ldots, z_{t-1}) = \int q_t(z|x, z_1, \ldots, z_{t-1})\, dP_k(x)$$

for the conditional density of the Z_t before ($k = 1$) and after ($k = 2$) the change point. To be specific, we let

$$P_1 = \mathrm{Unif}[0, 2\sigma], \quad P_2 = \kappa + \mathrm{Unif}[0, 2\sigma],$$

which satisfy that

$$\|P_1 - P_2\|_{\mathrm{TV}} = \frac{\kappa}{2\sigma}, \tag{S.24}$$

assuming that $\kappa < 2\sigma$.

For any $\nu \geq 1$ and $n \geq \nu$, we have that for any $n \geq \Delta$, it holds that

$$\frac{dP^n_{\kappa,\sigma,\nu}}{dP^n_{\kappa,\sigma,\infty}} = \exp\left(\sum_{i=\nu+1}^n W_i \right),$$

where $P_{\kappa,\sigma,\infty}$ indicates the joint distribution under which there is no change point and

$$W_i = \log \frac{m_{i,2}(Z_i|Z_{i-1}, \ldots, Z_1)}{m_{i,1}(Z_i|Z_{i-1}, \ldots, Z_1)}.$$

For any $\nu \geq 1$, define the event

$$\mathcal{E}_\nu = \left\{ \nu < T < \nu + \frac{c\sigma^2\alpha^{-2}\log(1/\gamma)}{\kappa^2},\ \sum_{i=\nu+1}^T W_i < \frac{3}{4}\log\left(\frac{1}{\alpha}\right) \right\}.$$

Then we have

$$\begin{aligned}\mathbb{P}_{\kappa,\sigma,\nu}(\mathcal{E}_\nu) &= \int_{\mathcal{E}_\nu} \exp\left(\sum_{i=\nu+1}^T W_i \right) dP_{\kappa,\sigma,\infty} \leq \exp\{(3/4)\log(1/\gamma)\}\, \mathbb{P}_{\kappa,\sigma,\infty}(\mathcal{E}_\nu) \\ &\leq \exp\{(3/4)\log(1/\gamma)\}\, \mathbb{P}_{\kappa,\sigma,\infty}\left\{ \nu < T < \nu + \frac{c\sigma^2\alpha^{-2}\log(1/\gamma)}{\kappa^2} \right\} \leq \gamma^{-3/4}\gamma = \gamma^{1/4}, \end{aligned} \tag{S.25}$$

where the first two inequalities follow from the definition of $\mathcal{E}_\nu$, and the last inequality follows from the definition of $\mathcal{D}(\gamma)$.

Step 2. Note that for any $i, z, z_{i-1}, \ldots, z_1$ and an arbitrary x_0 we have

$$\frac{m_{i,2}(z|z_{i-1}, \ldots, z_1)}{m_{i,1}(z|z_{i-1}, \ldots, z_1)} = \frac{\int q_{i,1}(z|x, z_{i-1}, \ldots, z_1)\, dP_2(x)}{\int q_{i,2}(z|x, z_{i-1}, \ldots, z_1)\, dP_1(x)} \leq \frac{e^\alpha q(z|x_0, z_{i-1}, \ldots, z_1) \int dP_2(x)}{e^{-\alpha} q(z|x_0, z_{i-1}, \ldots, z_1) \int dP_1(x)} = e^{2\alpha},$$

and we can similarly see that $m_{i,2}(z|z_{i-1}, \ldots, z_1)/m_{i,1}(z|z_{i-1}, \ldots, z_1) \geq e^{-2\alpha}$. When α is small, therefore, (say $\alpha \leq 0.5\log(0.5)$), we will have for any $z_1, \ldots, z_{i-1}$ that

$$\int m_{i,2}(z|z_{i-1}, \ldots, z_1) \log \frac{m_{i,2}(z|z_{i-1}, \ldots, z_1)}{m_{i,1}(z|z_{i-1}, \ldots, z_1)}\, dz$$

$$
\leq -\int m_{i,2}(z|z_{i-1},\ldots,z_1)\bigg\{\frac{m_{i,1}(z|z_{i-1},\ldots,z_1)}{m_{i,2}(z|z_{i-1},\ldots,z_1)} - 1 - \bigg(\frac{m_{i,1}(z|z_{i-1},\ldots,z_1)}{m_{i,2}(z|z_{i-1},\ldots,z_1)} - 1\bigg)^2\bigg\}\,dz
$$
$$
= \int \frac{\{m_{i,2}(z|z_{i-1},\ldots,z_1) - m_{i,1}(z|z_{i-1},\ldots,z_1)\}^2}{m_{i,2}(z|z_{i-1},\ldots,z_1)}\,dz \leq \min(4, e^{2\alpha})(e^{\alpha}-1)^2\|P_1 - P_2\|_{\mathrm{TV}}^2,
$$

where the final inequality is Lemma 1 in [2, Supplementary material]. Calculations around Lemma 1 in [2, Supplementary material] also reveal that

$$
\left|\log \frac{m_{i,2}(z|z_{i-1},\ldots,z_1)}{m_{i,1}(z|z_{i-1},\ldots,z_1)}\right| \leq \min(2, e^{\alpha})(e^{\alpha}-1)\|P_1 - P_2\|_{\mathrm{TV}}.
$$

It follows from the Azuma–Hoeffding inequality [e.g. 3, Corollary 2.20] that for any $x > 0$ and $t \in \mathbb{N}$ we have

$$
\begin{aligned}
&\mathbb{P}\bigg(\sum_{i=\nu+1}^{\nu+t} W_i \geq x + t\min(4, e^{2\alpha})(e^{\alpha}-1)^2\|P_1 - P_2\|_{\mathrm{TV}}^2 \,\bigg|\, Z_\nu,\ldots,Z_1\bigg) \\
&\leq \mathbb{P}\bigg(\sum_{i=\nu+1}^{\nu+t} \{W_i - \mathbb{E}(W_i|Z_{i-1},\ldots,Z_1)\} \geq x \,\bigg|\, Z_\nu,\ldots,Z_1\bigg) \\
&\leq \exp\bigg(-\frac{2x^2}{t\min(4, e^{2\alpha})(e^{\alpha}-1)^2\|P_1 - P_2\|_{\mathrm{TV}}^2}\bigg)
\end{aligned}
$$

Due to (S.24) and our assumption that $\alpha \leq 1$, for small enough $c > 0$ we have that

$$
\frac{c\sigma^2\alpha^{-2}\log(1/\gamma)}{\kappa^2} \times \min(4, e^{2\alpha})(e^{\alpha}-1)^2\|P_1 - P_2\|_{\mathrm{TV}}^2 \leq \frac{1}{4}\log(1/\gamma).
$$

For any $\nu \geq 1$ and $T \in \mathcal{D}(\gamma)$, since $\{T \geq \nu\} \in \mathcal{F}_{\nu-1}$, it therefore follows that for such c we have

$$
\begin{aligned}
&\mathbb{P}_{\kappa,\sigma,\nu}\left\{\nu < T < \nu + \frac{c\sigma^2\alpha^{-2}\log(1/\gamma)}{\kappa^2}, \sum_{i=\nu+1}^{T} W_i \geq (3/4)\log(1/\gamma) \,\Big|\, T > \nu\right\} \\
&\leq \operatorname{ess\,sup} \mathbb{P}_{\kappa,\sigma,\nu}\left\{\max_{1\leq t\leq \frac{c\sigma^2\alpha^{-2}\log(1/\gamma)}{\kappa^2}} \sum_{i=\nu+1}^{\nu+t} W_i \geq (3/4)\log(1/\gamma) \,\Big|\, Z_1,\ldots,Z_\nu\right\} \\
&\leq \operatorname{ess\,sup} \mathbb{P}_{\kappa,\sigma,\nu}\left[\max_{1\leq t\leq \frac{c\sigma^2\alpha^{-2}\log(1/\gamma)}{\kappa^2}} \sum_{i=\nu+1}^{\nu+t} \{W_i - \mathbb{E}(W_i|Z_{i-1},\ldots,Z_1)\} \geq (1/2)\log(1/\gamma) \Big| Z_1,\ldots,Z_\nu\right] \\
&\leq \frac{c\sigma^2\alpha^{-2}\log(1/\gamma)}{\kappa^2}\exp\left\{-\frac{(1/2)\log^2(1/\gamma)}{\frac{c\sigma^2\alpha^{-2}\log(1/\gamma)}{\kappa^2}\min(4, e^{2\alpha})(e^{\alpha}-1)^2\|P_1 - P_2\|_{\mathrm{TV}}^2}\right\} \\
&\leq \frac{c\sigma^2\alpha^{-2}\log(1/\gamma)}{\kappa^2}\exp\{-\log(1/\gamma)\} \leq \gamma^{1/4},
\end{aligned}
$$

where the third inequality follows by a union bound argument, the fourth inequality holds for small enough $c > 0$, and the last inequality holds for small enough γ. Since the upper bound is independent of ν, it holds that

$$
\sup_{\nu\geq 1} \mathbb{P}_{\kappa,\sigma,\nu}\left\{\nu < T < \nu + \frac{c\sigma^2\alpha^{-2}\log(1/\gamma)}{\kappa^2}, \sum_{i=\nu+1}^{T} W_i \geq (3/4)\log(1/\gamma) \,\Big|\, T \geq \nu\right\} \leq \gamma^{1/4},
$$

which leads to

$$
\sup_{\nu\geq 1} \mathbb{P}_{\kappa,\sigma,\nu}\left\{\nu < T < \nu + \frac{c\sigma^2\alpha^{-2}\log(1/\gamma)}{\kappa^2}, \sum_{i=\nu+1}^{T} W_i \geq (3/4)\log(1/\gamma)\right\} \leq \gamma^{1/4}. \tag{S.26}
$$

Combining (S.25) and (S.26), we have

$$
\sup_{\nu\geq 1} \mathbb{P}_{\kappa,\sigma,\nu}\left\{\nu < T < \nu + \frac{c\sigma^2\alpha^{-2}\log(1/\gamma)}{\kappa^2}\right\} \leq 2\alpha^{1/4}. \tag{S.27}
$$

Step 3. We now have, for any change point time Δ,

$$\begin{aligned}
&\mathbb{E}_{\kappa,\sigma,\Delta}\left\{(T-\Delta)_+\right\} \geq \frac{c\sigma^2\alpha^{-2}\log(1/\gamma)}{\kappa^2}\mathbb{P}_{\kappa,\sigma,\Delta}\left\{T-\Delta \geq \frac{c\sigma^2\alpha^{-2}\log(1/\gamma)}{\kappa^2}\right\} \\
=&\frac{c\sigma^2\alpha^{-2}\log(1/\gamma)}{\kappa^2}\left[\mathbb{P}_{\kappa,\sigma,\Delta}\{T>\Delta\} - \mathbb{P}_{\kappa,\sigma,\Delta}\left\{\Delta < T < \Delta + \frac{c\sigma^2\alpha^{-2}\log(1/\gamma)}{\kappa^2}\right\}\right] \\
\geq&\frac{c\sigma^2\alpha^{-2}\log(1/\gamma)}{2\kappa^2},
\end{aligned}$$

where the first inequality is due to Markov's inequality, the second is due to (S.27) and the definition of the class of $\mathcal{D}(\gamma)$ of stopping times. □

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