A Proof of Theorem 1

Recall that under maximum entropy RL, the Q-function is defined as $Q^{\pi}_{\text{ent}}(x_0, a_0) := \mathbb{E}_{\pi}[r_0 + \sum_{t=1}^{\infty} \gamma^t (r_t + c\mathcal{H}^{\pi}(x_t))]$ where $\mathcal{H}^{\mu}(x_t)$ is the entropy of the distribution $\pi^{\mu}(\cdot \mid x_t)$. The Bellman equation for Q-function is naturally

$$Q^{\pi}(x_0, a_0) = \mathbb{E}_{\pi}[r_0 + \gamma c\mathcal{H}^{\pi}(x_1) + \gamma Q^{\pi}(x_1, a_1)].$$

Let the optimal policy be π^*_{ent}. The relationship between the optimal policy and its Q-function is $\pi^*_{\text{ent}}(a \mid x) \propto \exp(Q^{\pi^*_{\text{ent}}}_{\text{ent}}(x, a)/c)$. We seek to establish $Q^{\pi^*_{\text{ent}}}_{\text{ent}}(x_0, a_0) \geq \mathbb{E}_{\mu}[r_0 + \gamma c\mathcal{H}^{\mu}(x_1) + \sum_{t=1}^{T-1} \gamma^t (r_t + c\mathcal{H}^{\mu}(x_{t+1})) + \gamma^T Q^{\pi}_{\text{ent}}(x_T, a_T)]$ for any policy μ, π.

We prove the results using induction. For the base case $T = 1$,

$$\begin{aligned}
Q^{\pi^*_{\text{ent}}}_{\text{ent}}(x_0, a_0) &= \mathbb{E}_{\pi^*_{\text{ent}}}[r_0 + \gamma c\mathcal{H}^{\pi_{\text{ent}}*}(x_1) + \gamma Q^{\pi^*_{\text{ent}}}_{\text{ent}}(x_1, a_1)] \\
&= \mathbb{E}_{x_1 \sim p(\cdot|x_0,a_0)}\big[r_0 + \gamma c\mathcal{H}^{\pi_{\text{ent}}*}(x_1) + \gamma \mathbb{E}_{\pi^*_{\text{ent}}}[Q^{\pi^*_{\text{ent}}}_{\text{ent}}(x_1, a_1)]\big] \\
&\geq \mathbb{E}_{x_1 \sim p(\cdot|x_0,a_0)}\big[r_0 + \gamma c\mathcal{H}^{\mu}(x_1) + \gamma \mathbb{E}_{\mu}[Q^{\pi^*_{\text{ent}}}_{\text{ent}}(x_1, a_1)]\big] \\
&\geq \mathbb{E}_{x_1 \sim p(\cdot|x_0,a_0)}\big[r_0 + \gamma c\mathcal{H}^{\mu}(x_1) + \gamma \mathbb{E}_{\mu}[Q^{\pi}_{\text{ent}}(x_1, a_1)]\big].
\end{aligned} \tag{10}$$

In the above, to make the derivations clear, we single out the reward r_0 and state $x_1 \sim p(\cdot \mid x_0, a_0)$, note that the distributions of these two quantities do not depend on the policy. The first inequality follows from the fact that $\pi^*_{\text{ent}}(\cdot \mid x) = \arg\max_{\pi}[c\mathcal{H}^{\pi}(x) + \mathbb{E}_{a \sim \pi(\cdot|x)} Q^{\pi_{\text{ent}}}_{\text{ent}}(x, a)]$. The second inequality follows from $Q^{\pi^*_{\text{ent}}}_{\text{ent}}(x, a) \geq Q^{\pi}_{\text{ent}}(x, a)$ for any policy π.

With the base case in place, assume that the result holds for $T \leq k - 1$. Consider the case $T = k$

$$\begin{aligned}
\mathbb{E}_{\mu}[r_0 + \gamma c\mathcal{H}^{\mu}(x_1) + \sum_{t=1}^{T-1} \gamma^t (r_t + c\mathcal{H}^{\mu}(x_{t+1})) + \gamma^T Q^{\pi}_{\text{ent}}(x_T, a_T)] \\
\leq \mathbb{E}_{\mu}\big[r_0 + \gamma c\mathcal{H}^{\mu}(x_1) + \gamma \mathbb{E}_{\mu}[Q^{\pi^*_{\text{ent}}}_{\text{ent}}(x_1, a_1)]\big] \\
\leq Q^{\pi^*_{\text{ent}}}_{\text{ent}}(x_0, a_0),
\end{aligned}$$

When $\pi = \mu$ we have the special case $\mathbb{E}_{\mu}[\sum_{t=0}^{\infty} \gamma^t r_t] \leq V^{\pi^*}(x_0)$, the lower bound which motivated the original lower-bound Q-learning based self-imitation learning [8].

B Proof of Theorem 2

For notational simplicity, let $\mathcal{U} := (\mathcal{T}^{\mu})^{n-1}\mathcal{T}^{\pi}$ and let $\tilde{\mathcal{U}}Q(x, a) := Q(x, a) + [UQ(x, a) - Q(x, a)]_+$. As a result, we could write $\mathcal{T}^{\alpha,\beta}_{\text{n,sil}} = (1 - \beta)\mathcal{T}^{\pi} + (1 - \alpha)\beta\tilde{\mathcal{U}} + \alpha\beta\mathcal{U}$.

First, we prove the contraction properties of $\mathcal{T}^{\mu}_{\beta,\text{n,sil}}$. Note that by construction $|\tilde{\mathcal{U}}Q_1(x, a) - \tilde{\mathcal{U}}Q_2(x, a)| \leq \max(|Q_1(x, a) - Q_2(x, a)|, |\mathcal{U}Q_1(x, a) - \mathcal{U}Q_2(x, a)|) \leq \| Q_1 - Q_2 \|_{\infty}$. Then through the triangle inequality, $\| \mathcal{T}^{\alpha,\beta}_{\text{n,sil}} Q_1 - \mathcal{T}^{\alpha,\beta}_{\text{n,sil}} Q_2 \|_{\infty} \leq (1 - \beta) \| \mathcal{T}^{\pi}Q_1 - \mathcal{T}^{\pi}Q_2 \|_{\infty} + (1 - \alpha)\beta \| \tilde{\mathcal{U}}Q_1 - \tilde{\mathcal{U}}Q_2 \|_{\infty} + \alpha\beta \| \mathcal{U}Q_1 - \mathcal{U}Q_2 \|_{\infty} \leq [(1-\beta)\gamma + (1-\alpha)\beta + \alpha\beta\gamma^n] \| Q_1 - Q_2 \|_{\infty}$. This proves the upper bound on the contraction rates of $\mathcal{T}^{\alpha,\beta}_{\text{n,sil}}$. Let $\eta(\alpha, \beta) = (1 - \beta)\gamma + (1 - \alpha)\beta + \alpha\beta\gamma^n$ and set $\eta(\alpha, \beta) < \gamma$, we deduce $\alpha > \frac{1-\gamma}{1-\gamma^n}$.

Next, we show properties of the fixed point $\tilde{Q}^{\alpha,\beta}$. This point uniquely exists because $\Gamma(\mathcal{T}^{\alpha,\beta}_{\text{n,sil}}) < 1$ if $(1 - \alpha)\beta < 1$. From $\mathcal{T}^{\alpha,\beta}_{\text{n,sil}} \tilde{Q}^{\alpha,\beta} = \tilde{Q}^{\alpha,\beta}$, we could derive by rearranging terms $(1 - \beta)(\mathcal{T}^{\pi}\tilde{Q} - \tilde{Q}) + \alpha\beta(\mathcal{U}\tilde{Q} - \tilde{Q}) = -(1 - \alpha)\beta(\tilde{\mathcal{U}}\tilde{Q} - \tilde{Q}) \leq 0$. This further implies that $\mathcal{T}^{\pi}\tilde{Q} \leq \tilde{Q}$. Now let $\mathcal{T} := \frac{(1-\beta)}{1-\beta+\alpha\beta}\mathcal{T}^{\pi} + \frac{\alpha\beta}{1-\beta+\alpha\beta}\mathcal{U}$. This simplifies to $\mathcal{T}\tilde{Q} - \tilde{Q} \leq 0$. By the monotonicity of $\mathcal{T}$, we see $Q^{t\pi + (1-t)\mu^{n-1}\pi} \geq \lim_{k \to} (\mathcal{T})^k \tilde{Q} = Q^{\pi}$ where $t = \frac{1-\beta}{1-\beta+\alpha\beta}$.

For the another set of inequalities, define $\tilde{H}Q := (1 - \beta)\mathcal{T}^* + (1 - \alpha)\beta\tilde{\mathcal{U}}Q + \alpha\beta(\mathcal{T}^*)^n$, where recall that $\mathcal{T}^*$ is the optimality Bellman operator.

First, note $\tilde{H}$ has Q^{π^*} as its unique fixed point. To see why, let $\tilde{Q}$ be a generic fixed point of $\tilde{H}$ such that $\tilde{H}\tilde{Q} = \tilde{Q}$. By rearranging terms, it follows that $(1-\beta)(\mathcal{T}^*\tilde{Q} - \tilde{Q}) + \alpha\beta((\mathcal{T}^*)^n\tilde{Q} - \tilde{Q}) = -(1-\alpha)\beta(\tilde{\mathcal{U}}\tilde{Q} - \tilde{Q}) \leq 0$. However, by construction $(\mathcal{T}^*)^i Q \geq Q, \forall i \geq 1, \forall Q$. This implies that $(1-\beta)(\mathcal{T}^*\tilde{Q} - \tilde{Q}) + \alpha\beta((\mathcal{T}^*)^n\tilde{Q} - \tilde{Q}) \geq 0$. As a result, $(1-\beta)(\mathcal{T}^*\tilde{Q} - \tilde{Q}) + \alpha\beta((\mathcal{T}^*)^n\tilde{Q} - \tilde{Q}) = 0$ and $\tilde{Q}$ is a fixed point of $t\mathcal{T}^* + (1-t)(\mathcal{T}^*)^n$. Since $t\mathcal{T}^* + (1-t)(\mathcal{T}^*)^n$ is strictly contractive as $\Gamma(t\mathcal{T}^* + (1-t)(\mathcal{T}^*)^n) \leq t\gamma + (1-t)\gamma^n \leq \gamma < 1$, its fixed point is unique. It is straightforward to deduce that Q^{π^*} is a fixed point of $t\mathcal{T}^* + (1-t)(\mathcal{T}^*)^n$ and we conclude that the only possible fixed point of $\tilde{H}$ is $\tilde{Q} = Q^{\pi^*}$. Finally, recall that by construction $\tilde{H}Q \geq Q, \forall Q$. By monotonicity, $Q^{\pi^*} = \lim_{k\rightarrow\infty}(\tilde{H})^k \tilde{Q}^{\alpha,\beta} \geq \tilde{Q}^{\alpha,\beta}$. In conclusion, we have shown $Q^{t\pi + (1-t)\mu^{n-1}\mu} \leq \tilde{Q}^{\alpha,\beta} \leq Q^{\pi^*}$.

C Additional theoretical results

Theorem 3. *Let π^* be the optimal policy and V^{π^*} its value function under standard* RL *formulation. Given a partial trajectory $(x_t, a_t)_{t=0}^n$, the following inequality holds for any n,*

$$V^{\pi^*}(x_0) \geq \mathbb{E}_\mu[\sum_{t=0}^{n-1} \gamma^t r_t + \gamma^n V^\pi(x_k)] \tag{11}$$

Proof. Let π, μ be any policy and π^* the optimal policy. We seek to show $V^{\pi^*}(x_0) \geq \mathbb{E}_\mu[\sum_{t=0}^{T-1} \gamma^t r_t + \gamma^T V^\pi(x_T)]$ for any $T \geq 1$.

We prove the results using induction. For the base case $T = 1$, $V^{\pi^*}(x_0) = \mathbb{E}_{\pi^*}[Q^{\pi^*}(x_0, a_0)] \geq \mathbb{E}_\mu[Q^{\pi^*}(x_0, a_0)] = \mathbb{E}_\mu[r_0 + \gamma V^{\pi^*}(x_1)] \geq \mathbb{E}_\mu[r_0 + \gamma V^\pi(x_1)]$, where the first inequality comes from the fact that $\pi^*(\cdot \mid x_0) = \arg\max_a Q^{\pi^*}(x_0, a)$. Now assume that the statement holds for any $T \leq k-1$, we proceed to the case $T = k$.

$$\begin{aligned}
\mathbb{E}_\mu[\sum_{t=0}^{k-1} \gamma^t r_t + \gamma^k V^\pi(x_k)] &= \mathbb{E}_\mu\big[r_0 + \gamma \mathbb{E}_\mu[\sum_{t=0}^{k-2} \gamma^t r_t + \gamma^{k-1} V^\pi(x_k)]\big] \\
&\leq \mathbb{E}_\mu[r_0 + \gamma V^{\pi^*}(x_1)] \leq V^{\pi^*}(x_0),
\end{aligned}$$

where the first inequality comes from the induction hypothesis and the second inequality follows naturally from the base case. This implies that n-step quantities of the form $V^{\pi^*}(x_0) \geq \mathbb{E}_\mu[\sum_{t=0}^{n-1} \gamma^t r_t + \gamma^n V^\pi(x_T)]$ are lower bounds of the optimal value function $V^{\pi^*}(x_0)$ for any $n \geq 1$. ☐

D Experiment details

Implementation details. The algorithmic baselines for deterministic actor-critic (TD3 and DDPG) are based on OpenAI Spinning Up https://github.com/openai/spinningup [51]. The baselines for stochastic actor-critic is based on PPO [18] and SIL+PPO [8] are based on the author code base https://github.com/junhyukoh/self-imitation-learning. Throughout the experiments, all optimizations are carried out via Adam optimizer [52].

Architecture. Deterministic actor-critic baselines, including TD3 and DDPG share the same network architecture following [51]. The Q-function network $Q_\theta(x, a)$ and policy $\pi_\phi(x)$ are both 2-layer neural network with $h = 300$ hidden units per layer, before the output layer. Hidden layers are interleaved with relu(x) activation functions. For the policy $\pi_\phi(x)$, the output is stacked with a tanh(x) function to ensure that the output action is in $[-1, 1]$. All baselines are run with default hyper-parameters from the code base.

Stochastic actor-critic baselines (e.g. PPO) implement value function $V_\theta(x)$ and policy $\pi_\phi(a \mid x)$ both as 2-layer neural network with $h = 64$ hidden units per layer and tanh activation. The stochastic policy $\pi_\phi(a \mid x)$ is a Gaussian $a \sim \mathcal{N}(\mu_\phi(x), \sigma^2)$ with state-dependent mean $\mu_\phi(x)$ and a global variance parameter σ^2. Other missing hyper-parameters take default values from the code base.

Table 1: Summary of the performance of algorithmic variants across benchmark tasks. We use *uncorrected* to denote prioritized sampling without IS corrections. Return-based SIL is represented as SIL with $n = \infty$. For each task, algorithmic variants with top performance are highlighted (two are highlighted if they are not statistically significantly different). Each entry shows mean $\pm$ std performance.

Tasks	**SIL $n = 5$**	**SIL $n = 5$ (uncorrected)**	**SIL $n = 1$ (uncorrected)**	**5-step**	**1-step**	**SIL $n = \infty$**
DMWALKERRUN	$\mathbf{642 \pm 107}$	$\mathbf{675 \pm 15}$	500 ± 138	246 ± 49	274 ± 100	320 ± 111
DMWALKERSTAND	$\mathbf{979 \pm 2}$	947 ± 18	899 ± 55	749 ± 150	487 ± 177	748 ± 143
DMWALKERWALK	731 ± 151	622 ± 197	601 ± 108	$\mathbf{925 \pm 10}$	793 ± 121	398 ± 203
DMCHEETAHRUN	$\mathbf{830 \pm 36}$	597 ± 64	702 ± 72	553 ± 92	643 ± 83	655 ± 59
ANT	$\mathbf{4123 \pm 364}$	3059 ± 360	$\mathbf{3166 \pm 390}$	1058 ± 281	3968 ± 401	3787 ± 411
HALFCHEETAH	8246 ± 784	9976 ± 252	$\mathbf{10417 \pm 364}$	6178 ± 151	$\mathbf{10100 \pm 481}$	8389 ± 386
ANT(B)	$\mathbf{2954 \pm 54}$	1690 ± 564	1851 ± 416	$\mathbf{2920 \pm 84}$	1866 ± 623	1884 ± 631
HALFCHEETAH(B)	$\mathbf{2619 \pm 129}$	$\mathbf{2521 \pm 128}$	2420 ± 109	1454 ± 338	2544 ± 31	2014 ± 378

D.1 Further implementation and hyper-parameter details

Generalized SIL for deterministic actor-critic. We adopt TD3 [27] as the baseline for deterministic actor-critic. TD3 maintains a Q-function network $Q_\theta(x,a)$ and a deterministic policy network $\pi_\theta(x)$ with parameter θ. The SIL subroutines adopt a prioritized experience replay buffer: the return-based SIL samples tuples according to the priority $[R^\mu(x,a) - Q_\theta(x,a)]_+$ and minimizes the loss function $[R^\mu(x,a) - Q_\theta(x,a)]_+$; the generalized SIL samples tuples according to the priority $[L^{\pi,\mu,n}(x,a) - Q_\theta(x,a)]_+$ and minimizes the loss function $[L^{\pi,\mu,n}(x,a) - Q_\theta(x,a)]_+$. The experience replay adopts the parameter $\alpha = 0.6, \beta = 0.1$ [53]. Throughout the experiments, TD3-based algorithms all employ $\alpha = 10^{-3}$ for the network updates.

To calculate the update target $L^{\pi,\mu,n}(x_0,a_0) = \sum_{t=0}^{n-1} \gamma^t r_t + Q_{\theta'}(x_n, \pi_{\theta'}(x_n))$ with partial trajectory $(x_t, a_t, r_t)_{t=0}^n$ along with the target value network $Q_{\theta'}(x,a)$ and policy network $\pi_{\theta'}(x)$. The target network is slowly updated as $\theta' = \tau\theta' + (1-\tau)\theta$ where $\tau = 0.995$ [14].

Generalized SIL for stochastic actor-critic. We adopt PPO [18] as the baseline algorithm and implement modifications on top of the SIL author code base https://github.com/junhyukoh/self-imitation-learning as well as the original baseline code https://github.com/openai/baselines [54]. All PPO variants use the default learning rate $\alpha = 3 \cdot 10^{-4}$ for both actor $\pi_\theta(a \mid x)$ and critic $V_\theta(x)$. The SIL subroutines are implemented as a prioritized replay with $\alpha = 0.6, \beta = 0.1$. For other details of SIL in PPO, please refer to the SIL paper [8].

The only difference between generalized SIL and SIL lies in the implementation of the prioritized replay. SIL samples tuples according to the priority $[R^\mu(x,a) - V_\theta(x)]_+$ and minimize the SIL loss function $([R^\mu(x,a) - V_\theta(x)]_+)^2$ for the value function, and $-\log \pi_\theta(a \mid x)[R^\mu(x,a) - V_\theta(x)]_+$ for the policy. Generalized SIL samples tuples according to the priority $([L^{\pi,\mu,n}(x,a) - V_\theta(x)]_+)^2$, and minimize the loss $([L^{\pi,\mu,n}(x,a) - V_\theta(x)]_+)^2$ and $-\log \pi_\theta(a \mid x)[L^{\pi,\mu,n}(X,a) - V_\theta(x)]_+$ for the value function/policy respectively.

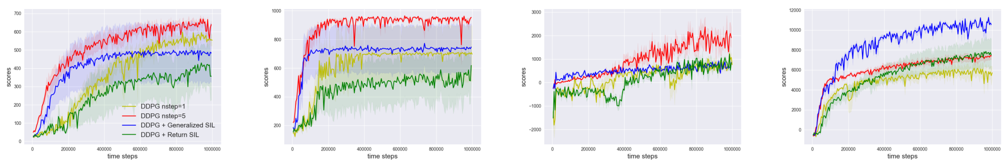
To calculate the update target $L^{\pi,\mu,n}(x_0,a_0) = \sum_{t=0}^{n-1} \gamma^t r_t + V_{\theta'}(x_n)$ with partial trajectory $(x_t, a_t, r_t)_{t=0}^n$ along with the target value network $V_{\theta'}(x)$. We apply the target network technique to stabilizie the update, where θ' is a delayed version of the major network θ and is updated as $\theta' = \tau\theta' + (1-\tau)\theta$ where $\tau = 0.995$.

D.2 Additional experiment results

Comparison across related baselines. We make clear the comparison between related baselines in Table 1. We present results for n-step TD3 with $n \in \{1, 5\}$; TD3 with generalized SIL with $n = 5$ and its variants with different setups for prioritized sampling; TD3 with return-based SIL ($n = \infty$). We show the results across all 8 tasks - in each entry of Table 1 we show the mean $\pm$ std of performance averaged over 3 seeds. The performance of each algorithmic variant is the average testing performance of the last 10^4 training steps (from a total of 10^6 training steps). The best

Table 2: Comparison between different replay schemes. For each task, algorithmic variants with top performance are highlighted (two are highlighted if they are not statistically significantly different). Each entry shows mean ± std performance.

Tasks	SIL $n = 5$	SIL $n = 5$ (uncorrected)	SIL $n = 5$ (no priority)
DMWALKERRUN	$\mathbf{642 \pm 107}$	$\mathbf{675 \pm 15}$	424 ± 127
DMWALKERSTAND	$\mathbf{979 \pm 2}$	947 ± 18	634 ± 184
DMWALKERWALK	$\mathbf{731 \pm 151}$	622 ± 197	$\mathbf{766 \pm 103}$
DMCHEETAHRUN	$\mathbf{830 \pm 36}$	597 ± 64	$\mathbf{505 \pm 182}$
ANT	$\mathbf{4123 \pm 364}$	3059 ± 360	4358 ± 496
HALFCHEETAH	8246 ± 784	$\mathbf{9976 \pm 252}$	8927 ± 596
ANT(B)	$\mathbf{2954 \pm 54}$	1690 ± 564	$\mathbf{2910 \pm 88}$
HALFCHEETAH(B)	$\mathbf{2619 \pm 129}$	$\mathbf{2521 \pm 128}$	2284 ± 85



(a) **DMWalkerRun** (b) **DMWalkerStand** (c) **Ant** (d) **HalfCheetah**

Figure 4: Standard evaluations on 4 simulation tasks for DDPG baselines. Different colors represent different algorithmic variants. Each curve shows the mean ± 0.5std of evaluation performance during training, averaged across 3 random seeds. The x-axis shows the time steps and the y-axis shows the cumulative returns.

algorithmic variant is highlighted in bold. We see that in general generalized SIL with $n = 5$ performs the best.

Ablation on the prioritized sampling. In prioritized sampling [53], when the tuples $d = (x_i, a_i, r_i)_{i=0}^n \in \mathcal{D}$ are sampled with priorities s_d, it is sampled with probability $p(d) \propto s_d^\alpha$. During updates, the IS correction consists in optimizing the loss $\mathbb{E}_d[w_d l_d]$ where l_d is the loss computed from tuple d and the IS correction weight $w_d = (N \cdot p_d)^{-\beta}$ where N is the number of tuples in the buffer $\mathcal{D}$.

We compare several prioritized sampling variants of generalized SIL in Table 2. There are three variants: SIL $n = 5$ with both prioritized sampling ($\alpha = 0.6$) and IS correction ($\beta = 0.1$); SIL $n = 5$ with prioritized sampling ($\alpha = 0.6$) only and without IS correction ($\beta = 0.0$); SIL $n = 5$ with no prioritized sampling ($\alpha = \beta = 0.0$). The performance setup in Table 2 is the same as in Table 1. It can be seen from Table 2 that generalized SIL performs the best with full prioritized sampling.

Results on DDPG. DDPG is a baseline actor-critic algorithm with a deterministic actor [15]. Compared to TD3, DDPG does not adopt a double-critic approach [27] and suffers from over-estimation bias of the Q-function [30].

We present the baseline evaluation result of DDPG in Figure 4, where we show the results for a few variants: DDPG with n-step update, $n \in \{1, 5\}$; DDPG with generalized SIL $n = 5$ and DDPG with return-based SIL ($n = \infty$). We see that the performance gains of DDPG with generalized SIL $n = 5$ are not as significant - indeed, overall DDPG with $n = 5$ has the best performance. We speculate that this is partly due to the over-estimation bias of DDPG: the formulation of generalized SIL is motivated by shifting the fixed point Q^π with an positive bias. The baseline algorithm benefits the most from generalized SIL when indeed in practice $Q_\theta \approx Q^\pi$. However, this is not the case for DDPG as the algorithm already has high positive bias in that $Q^\theta > Q^\pi$, which reduces the potential gains that come from generalized SIL.