8 Lemma 1 - Ergodic Markov Communication Graph

Lemma 1. *Let γ_0 be an initial state where all players only know their current reward. Define $\mathcal{S}$ as the set of all states that are reachable from γ_0 with positive probability after at least M transitions. The memory process $\Gamma(t)$ is an ergodic Markov Chain on $\mathcal{S}$. For a sufficiently large M and a proper communication protocol (Definition 4) there exists an $\varepsilon_0 > 0$ such that $\mathbb{P}_\pi\left(m \in \mathcal{U}_n^M(t)\right) \geq \varepsilon_0$ for each n, m, where π is the unique stationary distribution of $\Gamma(t)$.*

Proof. We emphasize that $\mathcal{S}$ is simply the set of "feasible states", where the values in $\mathcal{U}_n(t)$ do not contradict the graph process $G(t)$. The initial state γ_0 is the natural starting point for our algorithm, when players still have not learned anything about their peers and have received their own reward once in the past (so $t = 1$). Note that γ_0 is a transient state that is not in $\mathcal{S}$. Any state that the algorithm can encounter from $t = M + 1$ on, starting from γ_0, is in $\mathcal{S}$ by definition. Since the path starting from γ_0 is at least of length M, all the entries of $\mathcal{U}_n(t)$ are affected and the initial state γ_0 is remembered only through the sequence $G(t)$.

$\Gamma(t)$ on $\mathcal{S}$ is a Markov chain since $G(t)$ is a Markov chain, and given $G(t)$, the communication protocol is independent of $G(t-i)$ for $i \geq 1$ and is always independent of $\{\mathcal{U}_n(t-i)\}_{n=1}^N$ for $i \geq 1$. Old entries from more than M turns ago in $\{\mathcal{U}_n(t)\}_{n=1}^N$ are deterministically deleted.

We next show that $\Gamma(t)$ on $\mathcal{S}$ is an ergodic chain. We argue that from every $\gamma_1 \in \mathcal{S}$ there exists a positive probability path to any $\gamma_2 \in \mathcal{S}$. This path consists of:

- A path from γ_1 to a state γ_{G_0} for which $G(t) = G_0$, where G_0 is the graph in γ_0, and $\{\mathcal{U}_n(t)\}_{n=1}^N$ is arbitrary. This path must exist since $G(t)$ is an ergodic Markov chain that given $G(t-1)$, does not depend on $\{\mathcal{U}_n(t)\}_{n=1}^N$.
- A path from γ_{G_0} to γ_2. This path must exist since by definition of $\mathcal{S}$, there exists a path from γ_0 to γ_2. This path induces a sequence of graphs from G_0 to G_2 and broadcasting sets for each transition, that are constrained by $G(t)$. By repeating the same transitions, that are also available from γ_{G_0} since $G(t) = G_0$, this path leads to γ_2. This follows since $\mathcal{U}_n(t)$ has memory M, so all the remains of γ_1 are erased and replaced by these of γ_2. In fact, the exact same transitions lead to γ_2 from any state with $G(t) = G_0$.

This path can be made longer by just adding arbitrary transitions before step 1 above. Hence, there exists an $\tilde{L}$ such that for all $l \geq \tilde{L}$, a path of length l exists between γ_1 and γ_2, so $\Gamma(t)$ is aperiodic. We conclude that $\Gamma(t)$ on $\mathcal{S}$ is an ergodic Markov chain by definition.

Now we use the assumption that the graph union $\bigcup_{G \in \mathcal{G}} G$ is a connected graph, and that the communication protocol is proper. Since $\bigcup_{G \in \mathcal{G}} G$ is connected, there exists a path from any player m to player n such that each edge l on this path is in some $\tilde{G}_l \in \bigcup_{G \in \mathcal{G}} G$. Therefore, using the properness of the communication protocol, there is a positive probability that for any n, m, a message from player m can reach player n, containing the reward of player m (that is always known to it). In other words, for every m, n there exists a state $\gamma_m^n \in \mathcal{S}$ where $m \in \mathcal{U}_n^M(t)$ (so $(m, M) \in \mathcal{U}_n(t)$). This follows since this state is reachable from γ_0, by sequentially (but not successively) visiting all the graphs $\tilde{G}_1, \tilde{G}_2, \ldots$, propagating the message from m to n one step at a time. In all other times, where a non-relevant graph is visited, the time tag of the message τ just increases while the message "stays" at the same place. Such a path of graphs exists since $G(t)$ is an ergodic Markov chain. If M is large enough, this message will reach player n in $\tau \leq M$ steps, meaning that $m \in \mathcal{U}_n^\tau(t)$, which deterministically implies that $m \in \mathcal{U}_n^M(t)$, $M - \tau$ turns later. Hence, $\left\{\gamma \in \mathcal{S} \mid m \in \mathcal{U}_n^M(t)\right\}$ is non-empty and its stationary probability is positive for each n, m:

$$\pi_m^n \triangleq \mathbb{P}_\pi\left(m \in \mathcal{U}_n^M(t)\right) = \sum_{\gamma \in \left\{\gamma \in \mathcal{S} \mid m \in \mathcal{U}_n^M(t)\right\}} \pi_\gamma > 0. \tag{10}$$

□

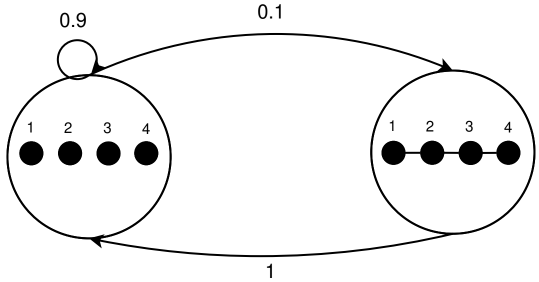

Figure 3: Communication graph process $G(t)$ for which $M = N = 4$ is not large enough

8.1 The Memory M

For a constant graph G, M only needs to be greater than the diameter of G. It is not possible to optimize the sum of rewards if the samples of one of the reward functions in the sum are always missing. Hence, an M larger than the diameter is also necessary for any algorithm to converge to an optimal action profile.

Even for a "well-behaved" Markov chain $G(t)$, requiring that M is larger than the average stationary diameter is enough. For instance, a well-behaved $G(t)$ changes only a limited number of edges each step, or changes all of them symmetrically (e.g., an i.i.d. $G(t)$ sequence). However, one can think of pathological chains where $G(t)$ changes very fast in an "adversarial way" to try to block certain messages from n to m from quickly reaching their destination. A simple example is given in Fig. 3, where Player 1 has to wait for at least 6 turns before a message from Player 4 can arrive. To keep our results general and simple, we choose to provide the weakest conditions possible on $G(t)$ and M. The fact that with less connected or less reliable communication graph the convergence time of Algorithm 1, or any other algorithm, is longer, is of course inevitable.

If the distribution of $G(t)$ is known to the designer that tunes the algorithm, then the required value for M can be computed. If the distribution of $G(t)$ is unknown, then an adaptive scheme is possible if the number of players N is known. In this adaptive scheme, each player holds a local version of M, denoted $M_n(t)$. Once in T_0 turns, player n increases $M_n(t)$ by one, until it observes that some reward value of player m was received sometime in the past, for all m. Simultaneously, the players can run a consensus algorithm on the maximum $M_n(t)$ to converge to a single value of M for all (which is in fact not necessary). This is summarized in the update rule:

$$M_n(t+1) = \max\left\{ M_n(t) + \mathbb{1}_{\{t \bmod T_0 = 0\}} \mathbb{1}_{\left\{\exists m,\, m \notin \bigcup_{\tau=1}^{t} \mathcal{U}_n(\tau)\right\}}, \max_{m \in \mathcal{N}_n} M_m(t) \right\}. \tag{11}$$

This update rule must converge since a large enough M exists such that with probability 1, eventually at some random t_0, $\bigcup_{\tau=1}^{t_0} \mathcal{U}_n(\tau) = \mathcal{N}$ for all n. Then, Theorem 1 can be applied with the initial conditions as given by the state of the system at time $t = t_0$.

In the special case of a constant G, the diameter of G is bounded by N so we can always use $M = N$. If N is unknown, it can be stochastically estimated using consensus algorithms, or alternatively the diameter of G can be directly estimated [55].

In terms of storage, it is important to note that the $M \times N$ memory table each player keeps is sparse, since each player receives only $O(\bar{d})$ reward values each turn, where $\bar{d}$ is the average degree of $G(t)$. For example, the diameter of a random Erdős–Rényi graph $G(N,p)$ is $O\left(\frac{\log N}{\log(Np)}\right)$ [56], resulting in space complexity of $O(\bar{d}M) = O\left(pN\frac{\log N}{\log(Np)}\right)$. For $p = \frac{c \log N}{N}$ with $c > 1$, for which $G(N,p)$ is connected with high probability, the space complexity is $O(\log^2 N)$. As a numerical example, in a large network G of $N = 10000$ players, average degree $\bar{d} = 100$ and diameter 100, even the most naive implementation that stores floating point numbers requires only 40KB.

9 Lemma 2 - Gradient Bias and Noise

Lemma 2 (Gradient Bias and Noise)**.** *Let $\delta(t) = \frac{\delta_0}{t^q}$ and $\eta(t) = \frac{\eta_0}{t^p}$ for some $\delta_0, \eta_0 > 0$, such that $0 < p, q \le 1$ and $p - q > \frac{1}{2}$ and define $\tau(t) \triangleq \sum_{i=1}^{t-1} \eta(i)$. For any real τ, let $\kappa(\tau)$ denote the unique value of t such that $\tau(t) \le \tau \le \tau(t+1)$. Define the filtration*[2]

$$\mathcal{F}_n(t) = \sigma\left(\left\{\boldsymbol{Z}_n(\tau - M), \boldsymbol{X}(\tau + 1 - M), a_m^n(\tau+1)\mathbb{1}_{\{m \in \mathcal{U}_n^M(\tau+1)\}}\right\}_{\tau=-1}^{t}\right). \tag{12}$$

Define the gradient bias of player n as

$$\beta_{g,n}(t) = \sum_{m=1}^{N} a_m^n(t) 1_{\{m \in \mathcal{U}_n^M(t)\}} \mathbb{E}\left\{\frac{d_n \boldsymbol{Z}_n(t-M)}{\delta(t-M)} u_m(t-M) \mid \mathcal{F}_n(t-1)\right\} - \sum_{m=1}^{N} a_m^n(t) 1_{\{m \in \mathcal{U}_n^M(t)\}} \nabla_{\boldsymbol{x}_n} u_m(\boldsymbol{X}(t-M)) \tag{13}$$

and the gradient noise of player n as

$$\xi_{g,n}(t) = \sum_{m=1}^{N} a_m^n(t) \mathbb{1}_{\{m \in \mathcal{U}_n^M(t)\}} \frac{d_n \boldsymbol{Z}_n(t-M)}{\delta(t-M)} u_m(t-M) - \sum_{m=1}^{N} a_m^n(t) \mathbb{1}_{\{m \in \mathcal{U}_n^M(t)\}} \mathbb{E}\left\{\frac{d_n \boldsymbol{Z}_n(t-M)}{\delta(t-M)} u_m(t-M) \mid \mathcal{F}_n(t-1)\right\}. \tag{14}$$

Then

1. *$\|\beta_{g,n}(t)\| \le O(\delta(t))$, with probability 1.*
2. *There exists a $T > 0$ such that for all $\mu > 0$:*

$$\lim_{j_0 \to \infty} \mathbb{P}\left(\sup_{j \ge j_0} \max_{0 \le \tau \le T} \left|\sum_{t=\kappa(jT)}^{\kappa(jT+\tau)-1} \eta(t) \xi_{g,n}(t)\right| \ge \mu\right) = 0. \tag{15}$$

Proof. Define $\boldsymbol{X}_n^{\delta}(t) \triangleq \left(1 - \frac{\delta(t)}{r}\right) \boldsymbol{X}_n(t) + \frac{\delta(t)\theta}{r}$, so $\tilde{\boldsymbol{X}}_n(t) = \boldsymbol{X}_n^{\delta}(t) + \delta(t) \boldsymbol{Z}_n(t)$. First note that, for each m, due to the Lipschitz continuity of $\nabla_{\boldsymbol{x}_n} u_m(\boldsymbol{x})$ with constant L we have

$$\left\|\nabla_{\boldsymbol{x}_n} u_m(\boldsymbol{X}(t)) - \nabla_{\boldsymbol{x}_n} u_m\left(\boldsymbol{X}^{\delta}(t)\right)\right\| \le L \left\|\boldsymbol{X}(t) - \boldsymbol{X}^{\delta}(t)\right\| = L \left\|\frac{\delta(t)}{r}(\boldsymbol{X}(t) - \theta)\right\| = L\delta(t) \left\|\frac{\boldsymbol{X}(t) - \theta}{r}\right\| = O(\delta(t)). \tag{16}$$

Now recall the definition of $\mathcal{F}_n(t)$ in (12). Then

$$\|\beta_{g,n}(t)\| = \left\|\sum_{m=1}^{N} a_m^n(t) 1_{\{m \in \mathcal{U}_n^M(t)\}} \left(\mathbb{E}\left\{\frac{d_n \boldsymbol{Z}_n(t-M)}{\delta(t-M)} u_m\left(\tilde{\boldsymbol{X}}(t-M)\right) \mid \mathcal{F}_n(t-1)\right\} - \nabla_{\boldsymbol{x}_n} u_m(\boldsymbol{X}(t-M))\right)\right\| \underset{(a)}{\le} \frac{1}{\varepsilon_0} \sum_{m=1}^{N} \left\|\mathbb{E}\left\{\frac{d_n \boldsymbol{Z}_n(t-M)}{\delta(t-M)} u_m\left(\tilde{\boldsymbol{X}}(t-M)\right) \mid \mathcal{F}_n(t-1)\right\} - \nabla_{\boldsymbol{x}_n} u_m(\boldsymbol{X}(t-M))\right\| \underset{(b)}{=} \frac{N}{\varepsilon_0} \left\|\nabla_{\boldsymbol{x}_n} u_m\left(\boldsymbol{X}^{\delta}(t-M)\right) - \nabla_{\boldsymbol{x}_n} u_m(\boldsymbol{X}(t-M))\right\| \underset{(c)}{=} O(\delta(t)) \tag{17}$$

[2] with the understanding that $\boldsymbol{Z}_n(t) = \boldsymbol{X}(t) = 0$ for $t \le 0$.

where (a) follows since $\left|a_m^n(t)\,\mathbb{1}_{\{m\in\mathcal{U}_n^M(t)\}}\right| \leq \frac{1}{\varepsilon_0}$ and (b) follows Lemma C.1 in [31], noting that $\boldsymbol{Z}_n(t-M)$ is independent of $\mathcal{F}_n(t-1)$ so the conditional expectation on $\mathcal{F}_n(t-1)$ averages over $\boldsymbol{Z}_n(t-M)$ (which also appears in $\tilde{\boldsymbol{X}}_n(t-M)$). Equality (c) follows from (16). Now note that $\xi_{g,n}(t)$ in (14) is bounded with probability 1, so both $\mathbb{E}\{|\xi_{g,n}(t)|\} < \infty$ and $\mathbb{E}\left\{|\xi_{g,n}(t)|^2\right\} < \infty$ for all t. Furthermore, $\xi_{g,n}(t)$ is a martingale difference sequence, since $\xi_{g,n}(t)$ is $\mathcal{F}_n(t)$-measurable and $\mathbb{E}\{\xi_{g,n}(t) \mid \mathcal{F}_n(t-1)\} = 0$. Therefore

$$Z(t) \triangleq \sum_{\tau=1}^{t} \eta(\tau)\,\xi_{g,n}(\tau) \tag{18}$$

is a martingale adapted to $\mathcal{F}_n(t)$. Hence, for every fixed a, $Z_a(t) \triangleq Z(t) - Z(a)$ is a martingale for $t \geq a$ and $Z_a^2(t)$ is a submartingale. Using Doob's inequality (see [57]) on the submartingale $Z_a^2(t)$, we obtain for any $\mu > 0$, for some constant $K > 0$ and any $a, b > 0$

$$\mathbb{P}\left(\max_{a\leq t\leq b}|Z_a(t)| \geq \mu\right) = \mathbb{P}\left(\max_{a\leq t\leq b}|Z(t)-Z(a)|^2 \geq \mu^2\right) \leq \frac{\mathbb{E}\left\{\left|\sum_{i=a+1}^{b}\eta(i)\,\xi_{g,n}(i)\right|^2\right\}}{\mu^2} \underset{(a)}{\leq} \frac{K\mathbb{E}\left\{\sum_{i=a+1}^{b}\frac{\eta^2(i)}{\delta^2(i)}\right\}}{\mu^2} \tag{19}$$

where (a) follows since $\mathbb{E}\left\{|\xi_{g,n}(t)|^2\right\} = O\left(\frac{1}{\delta^2(t)}\right)$ and for all $t_1 < t_2$

$$\mathbb{E}\{\xi_{g,n}(t_1)\,\xi_{g,n}(t_2)\} = \mathbb{E}\{\mathbb{E}\{\xi_{g,n}(t_1)\,\xi_{g,n}(t_2) \mid \mathcal{F}_n(t_1)\}\} = \mathbb{E}\{\xi_{g,n}(t_1)\,\mathbb{E}\{\xi_{g,n}(t_2) \mid \mathcal{F}_n(t_1)\}\} = 0. \tag{20}$$

Let $T > 0$. Now we add together terms in (19) where in the j-th term we choose $a = \kappa(jT)-1, b = \kappa(jT+T)-1$. Double counting to bound the possible overlaps, we obtain for any $\mu > 0$

$$\sum_{j=0}^{\infty}\mathbb{P}\left(\max_{0\leq\tau\leq T}\left|\sum_{t=\kappa(jT)}^{\kappa(jT+\tau)-1}\eta(t)\,\xi_{g,n}(t)\right| \geq \mu\right) \leq \frac{2K\mathbb{E}\left\{\sum_{t=1}^{\infty}\frac{\eta^2(t)}{\delta^2(t)}\right\}}{\mu^2} \underset{(a)}{<} \infty \tag{21}$$

where (a) follows since $p - q > \frac{1}{2}$ so $\sum_{t=1}^{\infty}\frac{\eta^2(t)}{\delta^2(t)} < \infty$. We conclude from the Borel-Cantelli Lemma [57] that for every $\mu > 0$

$$\lim_{j_0\to\infty}\mathbb{P}\left(\sup_{j\geq j_0}\max_{0\leq\tau\leq T}\left|\sum_{t=\kappa(jT)}^{\kappa(jT+\tau)-1}\eta(t)\,\xi_{g,n}(t)\right| \geq \mu\right) = 0. \tag{22}$$

□

10 Lemma 3 - Communication Bias and Noise

Lemma 3 (Communication Bias and Noise)**.** *Let $\delta(t) = \frac{\delta_0}{t^q}$ and $\eta(t) = \frac{\eta_0}{t^p}$ for some $\delta_0, \eta_0 > 0$, such that $0 < p, q \leq 1$ and $p > \frac{3+q}{4}$. Define $\tau(t) \triangleq \sum_{i=1}^{t-1}\eta(i)$. For any real τ, let $\kappa(\tau)$ denote the unique value of t such that $\tau(t) \leq \tau \leq \tau(t+1)$. Let π be the stationary distribution of $\Gamma(t)$ (Definition 5) for a large enough M, and define, for each n and m,*

$$\pi_m^n \triangleq \sum_{\gamma\in\{\gamma\in\mathcal{S}\,\mid\, m\in\mathcal{U}_n^M(t)\}}\pi_\gamma \tag{23}$$

which is the stationary probability that $m \in \mathcal{U}_n^M(t)$. Define the communication bias of player n by

$$\beta_{c,n}(t) = \sum_{m=1}^{N}\mathbb{1}_{\{m\in\mathcal{U}_n^M(\tau)\}}a_m^n(t)\,\nabla_{\boldsymbol{x}_n}u_m(\boldsymbol{X}(t-M)) - \sum_{m=1}^{N}\frac{\mathbb{1}_{\{m\in\mathcal{U}_n^M(t)\}}}{\pi_m^n}\nabla_{\boldsymbol{x}_n}u_m(\boldsymbol{X}(t-M)) \tag{24}$$

and the communication noise of player n by

$$\xi_{c,n}(t) = \sum_{m=1}^{N} \left(\frac{\mathbb{1}_{\{m \in \mathcal{U}_n^M(t)\}}}{\pi_m^n} - 1 \right) \nabla_{\boldsymbol{x}_n} u_m \left(\boldsymbol{X}(t-M) \right). \tag{25}$$

Then

1. *$\beta_{c,n}(t)$ is bounded and converges to zero with probability 1.*
2. *There exists $T > 0$ such that for all $\mu > 0$:*

$$\lim_{j_0 \to \infty} \mathbb{P} \left(\sup_{j \geq j_0} \max_{0 \leq \tau \leq T} \left| \sum_{t=\kappa(jT)}^{\kappa(jT+\tau)-1} \eta(t) \xi_{c,n}(t) \right| \geq \mu \right) = 0. \tag{26}$$

Proof. Let $\varepsilon > 0$. We start from the communication bias, which satisfies

$$\begin{aligned} \|\beta_{c,n}(t)\| &= \left\| \sum_{m=1}^{N} \left(a_m^n(t) \mathbb{1}_{\{m \in \mathcal{U}_n^M(t)\}} - \frac{\mathbb{1}_{\{m \in \mathcal{U}_n^M(t)\}}}{\pi_m^n} \right) \nabla_{\boldsymbol{x}_n} u_m \left(\boldsymbol{X}(t-M) \right) \right\| \underset{(a)}{\leq} \\ & L \sum_{m=1}^{N} \left| \mathbb{1}_{\{m \in \mathcal{U}_n^M(t)\}} \right| \left| a_m^n(t) - \frac{1}{\pi_m^n} \right| \underset{(b)}{\leq} L \sum_{m=1}^{N} \left| a_m^n(t) - \frac{1}{\pi_m^n} \right| = \\ & L \sum_{m=1}^{N} \left| \frac{\pi_m^n - \max\{p_m^n(t), \varepsilon_0\}}{\max\{p_m^n(t), \varepsilon_0\} \pi_m^n} \right| \underset{(c)}{\leq} L \sum_{m=1}^{N} \frac{1}{\delta_0 \varepsilon_0} \left| \pi_m^n - p_m^n(t) \right| \leq \frac{LN}{\delta_0 \varepsilon_0} \end{aligned} \tag{27}$$

where (a) uses the Lipschitz continuity of $u_m(\boldsymbol{x})$ with constant L, so $\|\nabla_{\boldsymbol{x}_n} u_m(\boldsymbol{X}(t-M))\| \leq L$. In (b) we used $\left|\mathbb{1}_{\{m \in \mathcal{U}_n^M(t)\}}\right| \leq 1$. Inequality (c) follows from $p_m^n(\tau) \geq \varepsilon_0$ and the assumption that $G(t)$ is an ergodic Markov chain for which the union of the support is a connected graph. Therefore, we know from Lemma 1 that for a large enough M, there must exist an $\delta_0 > 0$ such that $\pi_m^n \geq \delta_0$ for each n, m. Since $\pi_m^n \geq \delta_0 \geq \varepsilon_0$, then replacing $p_m^n(t)$ by ε_0 as the estimator in case that $p_m^n(t) < \varepsilon_0$ can only improve the estimation, i.e.,

$$\left| \pi_m^n - \max\{p_m^n(t), \varepsilon_0\} \right| \leq \left| \pi_m^n - p_m^n(t) \right|. \tag{28}$$

Inequality (27) shows that $\beta_{c,n}(t)$ is bounded with probability 1. We now proceed to show that $\beta_{c,n}(t)$ converges to zero with probability 1. Let $0 < \varepsilon \leq \delta_0$. By applying the tail bound for Markov chains from [54] with $\Delta = \frac{\varepsilon}{\pi_m^n} \leq 1$ and the function $f(\gamma(t)) = \mathbb{1}_{\{m \in \mathcal{U}_n^M(t)\}}$, we obtain for some constants $C_0, C > 0$

$$\mathbb{P}\left(\left| p_m^n(t) - \pi_m^n \right| \geq \varepsilon \right) = \mathbb{P}\left(\left| \sum_{\tau=1}^{t} \left(\mathbb{1}_{\{m \in \mathcal{U}_n^M(\tau)\}} - \pi_m^n t \right) \right| \geq \Delta \pi_m^n t \right) \leq \frac{C_0}{\sqrt{\pi_{\gamma_0}}} e^{-\frac{\varepsilon^2}{72 \pi_m^n T_m} t} \underset{(a)}{\leq} \frac{C}{t^2} \tag{29}$$

where T_m is the mixing time of $\mathit{\Gamma}(t)$ with accuracy $\frac{1}{8}$ and γ_0 is the initial state of $\Gamma(t)$. Inequality (a) follows for a sufficiently large t. Hence

$$\sum_{t=1}^{\infty} \mathbb{P}\left(\left| p_m^n(t) - \pi_m^n \right| \geq \varepsilon \right) \leq \sum_{t=1}^{\infty} \frac{C}{t^2} < \infty. \tag{30}$$

Using (30), we conclude from the Borel Cantelli Lemma (see [57]) that, for every $\varepsilon > 0$, the probability that $|p_m^n(t) - \pi_m^n| \geq \varepsilon$ infinitely often is zero, which shows that $|p_m^n(t) - \pi_m^n| \to 0$ with probability 1 as $t \to \infty$. From (27) this also shows that $\beta_{c,n}(t)$ converges to zero with probability 1.

Next we analyze the noise $\xi_{c,n}(t)$. Define $S(t)=\sum_{\tau=1}^{t}\xi_{c,n}(\tau)$. Let $b_{n,m}(\tau)=\frac{\mathbb{1}_{\{m\in\mathcal{U}_n^M(\tau)\}}}{\pi_m^n}-1$. We want to show that $\eta(t)S(t)\to 0$ with probability 1 as $t\to\infty$. For that purpose we write

$$\|\eta(t)S(t)\|=\eta(t)\left\|\sum_{\tau=1}^{t}\sum_{m=1}^{N}b_{n,m}(\tau)\nabla_{\boldsymbol{x_n}}u_m(\boldsymbol{X}(\tau-M))\right\|\leq$$
$$\eta(t)\sum_{i=1}^{\frac{t}{\lceil t^r\rceil}}\left\|\sum_{\tau=1+(i-1)\lceil t^r\rceil}^{i\lceil t^r\rceil}\sum_{m=1}^{N}b_{n,m}(\tau)\nabla_{\boldsymbol{x_n}}u_m(\boldsymbol{X}(\tau-M))\right\|\leq$$
$$\underbrace{\eta(t)\sum_{i=1}^{\frac{t}{\lceil t^r\rceil}}\left\|\sum_{\tau=1+(i-1)\lceil t^r\rceil}^{i\lceil t^r\rceil}\sum_{m=1}^{N}b_{n,m}(\tau)\nabla_{\boldsymbol{x_n}}u_m(\boldsymbol{X}(i\lceil t^r\rceil-M))\right\|}_{A}$$
$$\underbrace{+\eta(t)\sum_{i=1}^{\frac{t}{\lceil t^r\rceil}}\left\|\sum_{\tau=1+(i-1)\lceil t^r\rceil}^{i\lceil t^r\rceil}\sum_{m=1}^{N}b_{n,m}(\tau)\left(\nabla_{\boldsymbol{x_n}}u_m(\boldsymbol{X}(\tau-M))-\nabla_{\boldsymbol{x_n}}u_m(\boldsymbol{X}(i\lceil t^r\rceil-M))\right)\right\|}_{B}. \tag{31}$$

We start by showing that the A term in (31) converges to zero with probability 1. We have for each i

$$\eta(t)\sum_{i=1}^{\frac{t}{\lceil t^r\rceil}}\left\|\sum_{\tau=1+(i-1)\lceil t^r\rceil}^{i\lceil t^r\rceil}\sum_{m=1}^{N}\left(\frac{\mathbb{1}_{\{m\in\mathcal{U}_n^M(\tau)\}}}{\pi_m^n}-1\right)\nabla_{\boldsymbol{x_n}}u_m(\boldsymbol{X}(i\lceil t^r\rceil-M))\right\|=$$
$$\eta(t)\sum_{i=1}^{\frac{t}{\lceil t^r\rceil}}\left\|\sum_{m=1}^{N}\nabla_{\boldsymbol{x_n}}u_m(\boldsymbol{X}(i\lceil t^r\rceil-M))\sum_{\tau=1+(i-1)\lceil t^r\rceil}^{i\lceil t^r\rceil}\left(\frac{\mathbb{1}_{\{m\in\mathcal{U}_n^M(\tau)\}}}{\pi_m^n}-1\right)\right\|\leq$$
$$\eta(t)\sum_{i=1}^{\frac{t}{\lceil t^r\rceil}}\sum_{m=1}^{N}\|\nabla_{\boldsymbol{x_n}}u_m(\boldsymbol{X}(i\lceil t^r\rceil-M))\|\left|\sum_{\tau=1+(i-1)\lceil t^r\rceil}^{i\lceil t^r\rceil}\left(\frac{\mathbb{1}_{\{m\in\mathcal{U}_n^M(\tau)\}}}{\pi_m^n}-1\right)\right|\leq$$
$$L\sum_{m=1}^{N}\eta(t)\sum_{i=1}^{\frac{t}{\lceil t^r\rceil}}\left|\sum_{\tau=1+(i-1)\lceil t^r\rceil}^{i\lceil t^r\rceil}\left(\frac{\mathbb{1}_{\{m\in\mathcal{U}_n^M(\tau)\}}}{\pi_m^n}-1\right)\right|. \tag{32}$$

Let $\varepsilon>0$. For each n,m, we apply again the tail bound for Markov chains from [54], this time with $\Delta=\frac{\varepsilon}{t\eta(t)}$ and the function $f(\gamma(t))=\mathbb{1}_{\{m\in\mathcal{U}_n^M(t)\}}$ to obtain, for some constants $C_0,C>0$

$$\mathbb{P}\left(\eta(t)\left|\sum_{\tau=1+(i-1)\lceil t^r\rceil}^{i\lceil t^r\rceil}\left(\frac{\mathbb{1}_{\{m\in\mathcal{U}_n^M(\tau)\}}}{\pi_m^n}-1\right)\right|\geq\frac{\varepsilon}{\frac{t}{\lceil t^r\rceil}}\right)=$$
$$\mathbb{P}\left(\left|\sum_{\tau=1+(i-1)\lceil t^r\rceil}^{i\lceil t^r\rceil}\left(\mathbb{1}_{\{m\in\mathcal{U}_n^M(\tau)\}}-\pi_m^n\right)\right|\geq\frac{\varepsilon}{t\eta(t)}\pi_m^n\lceil t^r\rceil\right)\leq\frac{C_0}{\sqrt{\pi_{\gamma_0}}}e^{-\frac{\varepsilon^2\pi_m^n}{72T_m}\frac{t^r}{t^2\eta^2(t)}}\underset{(a)}{\leq}\frac{C}{t^3} \tag{33}$$

where (a) follows for a sufficiently large t, assuming $2p>2-r$, since $\frac{1}{t^{2-r}\eta^2(t)}=\frac{1}{\eta_0^2}t^{2p+r-2}$. We conclude from the union bound that

$$\sum_{t=1}^{\infty}\mathbb{P}\left(\eta(t)\sum_{i=1}^{\frac{t}{\lceil t^r\rceil}}\left|\sum_{\tau=1+(i-1)\lceil t^r\rceil}^{i\lceil t^r\rceil}\left(\frac{\mathbb{1}_{\{m\in\mathcal{U}_n^M(\tau)\}}}{\pi_m^n}-1\right)\right|\geq\varepsilon\right)\leq$$
$$\sum_{t=1}^{\infty}\left(\sum_{i=1}^{\frac{t}{\lceil t^r\rceil}}\mathbb{P}\left(\eta(t)\left|\sum_{\tau=1+(i-1)\lceil t^r\rceil}^{i\lceil t^r\rceil}\left(\frac{\mathbb{1}_{\{m\in\mathcal{U}_n^M(\tau)\}}}{\pi_m^n}-1\right)\right|\geq\frac{\varepsilon}{\frac{t}{\lceil t^r\rceil}}\right)\right)\leq\sum_{t=1}^{\infty}\frac{C}{t^{2+r}}<\infty \tag{34}$$

so from the Borel Cantelli Lemma (see [57]) we conclude that the probability that $\eta(t)\sum_{i=1}^{\frac{t}{\lceil t^r\rceil}}\left|\sum_{\tau=1+(i-1)\lceil t^r\rceil}^{i\lceil t^r\rceil}\left(\frac{\mathbb{1}_{\{m\in\mathcal{U}_n^M(\tau)\}}}{\pi_m^n}-1\right)\right|\geq\varepsilon$ infinitely often is zero. This shows that if $2p>2-r$ then with probability 1, as $t\to\infty$:

$$\eta(t)\sum_{i=1}^{\frac{t}{\lceil t^r\rceil}}\left|\sum_{\tau=1+(i-1)\lceil t^r\rceil}^{i\lceil t^r\rceil}\left(\frac{\mathbb{1}_{\{m\in\mathcal{U}_n^M(t)\}}}{\pi_m^n}-1\right)\right|\to 0. \tag{35}$$

Then, from (32) we see that the A term in (31) converges to zero in probability 1.

Next we show that the B term in (31) converges to zero with probability 1 as well. We have

$$\eta(t)\sum_{i=1}^{\frac{t}{\lceil t^r\rceil}}\left\|\sum_{\tau=1+(i-1)\lceil t^r\rceil}^{i\lceil t^r\rceil}\sum_{m=1}^{N}b_{n,m}(\tau)\left(\nabla_{\boldsymbol{x_n}}u_m\left(\boldsymbol{X}(\tau-M)\right)-\nabla_{\boldsymbol{x_n}}u_m\left(\boldsymbol{X}\left(i\lceil t^r\rceil-M\right)\right)\right)\right\|\underset{(a)}{\leq}$$
$$\frac{\eta(t)}{\varepsilon_0}\sum_{m=1}^{N}\sum_{i=1}^{\frac{t}{\lceil t^r\rceil}}\sum_{\tau=1+(i-1)\lceil t^r\rceil}^{i\lceil t^r\rceil}\left\|\nabla_{\boldsymbol{x_n}}u_m\left(\boldsymbol{X}(\tau-M)\right)-\nabla_{\boldsymbol{x_n}}u_m\left(\boldsymbol{X}\left(i\lceil t^r\rceil-M\right)\right)\right\|\underset{(b)}{\leq}$$
$$\eta(t)\frac{NL}{\varepsilon_0}\sum_{i=1}^{\frac{t}{\lceil t^r\rceil}}\sum_{\tau=1+(i-1)\lceil t^r\rceil}^{i\lceil t^r\rceil}\left\|\boldsymbol{X}(\tau-M)-\boldsymbol{X}\left(i\lceil t^r\rceil-M\right)\right\|\underset{(c)}{\leq}$$
$$\frac{4\left(N^2\max_n d_n\right)U_{\max}}{\varepsilon_0}\eta(t)\sum_{i=1}^{\frac{t}{\lceil t^r\rceil}}\lceil t^r\rceil\sum_{\tau=1+(i-1)\lceil t^r\rceil}^{i\lceil t^r\rceil}\frac{\eta(\tau)}{\delta(\tau)} \tag{36}$$

where in (a) we used $b_{n,m}(\tau)=\left|\frac{\mathbb{1}_{\{m\in\mathcal{U}_n^M(\tau)\}}}{\pi_m^n}-1\right|\leq\frac{1}{\varepsilon_0}$ and in (b) we used the Lipschitz continuity of $\nabla_{\boldsymbol{x}_n}u_m(\boldsymbol{x})$. In (c) we used that for all $1+(i-1)\lceil t^r\rceil\leq\tau\leq i\lceil t^r\rceil$

$$\left\|\boldsymbol{X}(\tau-M)-\boldsymbol{X}\left(i\lceil t^r\rceil-M\right)\right\|\leq\sum_{l=\tau}^{i\lceil t^r\rceil-1}\left\|\boldsymbol{X}(l-M)-\boldsymbol{X}(l+1-M)\right\|\underset{(a)}{\leq}$$
$$4\frac{\left(N^2\max_n d_n\right)U_{\max}}{\varepsilon_0}\sum_{l=\tau}^{i\lceil t^r\rceil-1}\frac{\eta(l)}{\delta(l)}\leq 4\frac{\left(N^2\max_n d_n\right)U_{\max}}{\varepsilon_0}\lceil t^r\rceil\frac{\eta(\tau)}{\delta(\tau)} \tag{37}$$

where in (a), we used the continuity of $u_m(\boldsymbol{x})$ on the compact set $\mathcal{A}_m$ and the triangle inequality, and defined $U_{\max}\triangleq\max_m\max_{\boldsymbol{x}\in\mathcal{A}_m}|u_m(\boldsymbol{x})|$. We note that (a) holds only for $t>M$, so $\frac{t+M}{t}\leq 2$ and $\frac{\eta(l)}{\delta(l-M)}\leq 2\frac{\eta(l)}{\delta(l)}$ since $q\leq 1$. Inequality (a) also uses that by definition of $\prod_{\mathcal{A}}$ we have $\left\|\boldsymbol{z}-\prod_{\mathcal{A}}(\boldsymbol{z})\right\|\leq\|\boldsymbol{z}-\boldsymbol{x}\|$ for every $\boldsymbol{x}\in\mathcal{A}$ and every $\boldsymbol{z}$, so:

$$\|\boldsymbol{X}(t+1)-\boldsymbol{X}(t)\|\leq$$
$$\left\|\boldsymbol{X}(t)+\eta(t)\boldsymbol{Y}(t)-\prod_{\mathcal{A}_1\times\ldots\times\mathcal{A}_N}\left(\boldsymbol{X}(t)+\eta(t)\boldsymbol{Y}(t)\right)\right\|+\|\eta(t)\boldsymbol{Y}(t)\|\leq 2\|\eta(t)\boldsymbol{Y}(t)\|. \tag{38}$$

Finally,

$$\eta(t)\sum_{i=1}^{\frac{t}{\lceil t^r\rceil}}\lceil t^r\rceil\sum_{\tau=1+(i-1)\lceil t^r\rceil}^{i\lceil t^r\rceil}\frac{\eta(\tau)}{\delta(\tau)}\leq\left(t^r+1\right)\eta(t)\sum_{\tau=1}^{t}\frac{\eta(\tau)}{\delta(\tau)}\underset{(a)}{=}O\left(t^{-2p+q+1+r}\right) \tag{39}$$

where (a) uses, that since $0<p,q\leq 1$ then

$$\sum_{\tau=1}^{t}\frac{\eta(\tau)}{\delta(\tau)}\leq\frac{\eta_0}{\delta_0}+\frac{\eta_0}{\delta_0}\int_1^t\tau^{-(p-q)}d\tau=\frac{\eta_0}{\delta_0}+\frac{\eta_0}{\delta_0}\left(\frac{t^{-(p-q)+1}}{1-p+q}-\frac{1}{1-p+q}\right)=O\left(t^{-p+q+1}\right). \tag{40}$$

We conclude that if $2p > q+1+r$, then the B term in (31) converges to zero in probability 1.

Hence, if both $2p > 2-r$ (for (33)) and $2p > q+1+r$ (for (39)) then by (31), $\eta(t)S(t) \to 0$ as $t \to \infty$ with probability 1. Optimizing over the requirements, we choose $r = \frac{1-q}{2}$ which leads to the condition $p > \frac{q+3}{4}$.

For the last step we use that $\eta(t)S(t) \to 0$ with probability 1 as $t \to \infty$ to show (26). The key is the following telescoping identity for any integers a, b:

$$\sum_{\tau=a}^{b} \eta(\tau)\xi_{c,n}(\tau) = \eta(b)(S(b) - S(a-1)) + \sum_{\tau=a}^{b-1} (S(\tau) - S(a-1))(\eta(\tau) - \eta(\tau+1)) \quad (41)$$

where for $a = 1$ and $b = \kappa(t) - 1$ we obtain

$$\sum_{\tau=1}^{\kappa(t)-1} \eta(\tau)\xi_{c,n}(\tau) = \eta(\kappa(t)-1)S(\kappa(t)-1) + \sum_{\tau=1}^{\kappa(t)-2} S(\tau)\eta(\tau)\frac{\eta(\tau) - \eta(\tau+1)}{\eta(\tau)}. \quad (42)$$

Let $T > 0$ and let $0 \le \tau \le T$. By definition of $\kappa(\tau)$, $\sum_{t=\kappa(jT)-1}^{\kappa(jT+\tau)-2} \eta(t) \to \tau$ as $j \to \infty$. Recall that since $\sum_{t=1}^{\infty} \eta(t) = \infty$ then $\kappa(jT) \to \infty$ as $j \to \infty$. Let $\mu > 0$. Hence, for almost every $\omega \in \Omega$, there exists a $J(\omega)$ (that depends only on μ, η_0, T and not on τ) such that for all $j > J(\omega)$ we have $|S(t)\eta(t)| \le \frac{\mu}{4}\min\left\{1, \frac{\eta_0}{T}\right\}$ for all $t \ge \kappa(jT) - 1$ and that $\sum_{t=\kappa(jT)-1}^{\kappa(jT+\tau)-2} \eta(t) < 2T$. Then for all $j > J(\omega)$ we also have

$$\begin{aligned}\left|\sum_{t=\kappa(jT)}^{\kappa(jT+\tau)-1} \eta(t)\xi_{c,n}(t)\right| &= \left|\sum_{t=1}^{\kappa(jT+\tau)-1} \eta(t)\xi_{c,n}(t) - \sum_{t=1}^{\kappa(jT)-1} \eta(t)\xi_{c,n}(t)\right| \underset{(a)}{\le} \\ &|\eta(\kappa(jT+\tau)-1)S(\kappa(jT+\tau)-1)| + |\eta(\kappa(jT)-1)S(\kappa(jT)-1)| \\ &+ \sum_{t=\kappa(jT)-1}^{\kappa(jT+\tau)-2} |S(t)\eta(t)| \left|\frac{\eta(t) - \eta(t+1)}{\eta(t)}\right| \underset{(b)}{\le} \frac{\mu}{2} + \frac{\mu}{4T}\sum_{t=\kappa(jT)-1}^{\kappa(jT+\tau)-2} \eta(t) < \mu \end{aligned} \quad (43)$$

where (a) uses (42) with the triangle inequality, and (b) follows since for all $j > J(\omega)$ we have $\sum_{t=\kappa(jT)}^{\kappa(jT+\tau)-1} \eta(t) < 2T$ and that for all $t \ge \kappa(jT)$

$$\left|\frac{\eta(t) - \eta(t+1)}{\eta(t)}\right| = \frac{\left|\frac{1}{t^p} - \frac{1}{(t+1)^p}\right|}{\frac{1}{t^p}} = \left|1 - \left(\frac{t}{1+t}\right)^p\right| \le \frac{1}{1+t} \le \frac{\eta(t)}{\eta_0}. \quad (44)$$

We conclude that for every $\mu > 0$

$$\lim_{j_0 \to \infty} \mathbb{P}\left(\sup_{j \ge j_0} \max_{0 \le \tau \le T} \left|\sum_{t=\kappa(jT)}^{\kappa(jT+\tau)-1} \eta(t)\xi_{c,n}(t)\right| \ge \mu\right) = 0. \quad (45)$$

□

11 Proof of Theorem 1

Using Lemma 2 and Lemma 3 to control the noises and biases, we can now prove Theorem 1.

Proof. First note that for every n

$$\sum_{m=1}^{N} a_m^n(t)\mathbb{1}_{\{m \in \mathcal{U}_n^M(t)\}}\left(\frac{d_n \boldsymbol{Z}_n(t-M)}{\delta(t-M)}u_m(t-M) - \nabla_{\boldsymbol{x_n}} u_m(\boldsymbol{X}(t-M))\right) = \xi_{g,n}(t) + \beta_{g,n}(t) \quad (46)$$

and

$$\sum_{m=1}^{N}\left(a_m^n(t)\,\mathbb{1}_{\{m\in\mathcal{U}_n^M(t)\}}-1\right)\nabla_{\boldsymbol{x_n}}u_m(\boldsymbol{X}(t-M))=\beta_{c,n}(t)+\xi_{c,n}(t). \quad (47)$$

So we can write

$$\boldsymbol{Y}_n(t)=\frac{d_n\boldsymbol{Z}_n(t-M)}{\delta(t-M)}\sum_{m=1}^{N}a_m^n(t)\,\mathbb{1}_{\{m\in\mathcal{U}_n^M(t)\}}u_m(t-M)=$$
$$\sum_{m=1}^{N}a_m^n(t)\,\mathbb{1}_{\{m\in\mathcal{U}_n^M(t)\}}\left(\frac{d_n\boldsymbol{Z}_n(t-M)}{\delta(t-M)}u_m(t-M)-\nabla_{\boldsymbol{x_n}}u_m(\boldsymbol{X}(t-M))\right)+$$
$$\sum_{m=1}^{N}a_m^n(t)\,\mathbb{1}_{\{m\in\mathcal{U}_n^M(t)\}}\nabla_{\boldsymbol{x_n}}u_m(\boldsymbol{X}(t-M))=$$
$$\xi_{g,n}(t)+\beta_{g,n}(t)+\sum_{m=1}^{N}\left(a_m^n(t)\,\mathbb{1}_{\{m\in\mathcal{U}_n^M(t)\}}-1\right)\nabla_{\boldsymbol{x_n}}u_m(\boldsymbol{X}(t-M))+g_n(\boldsymbol{X}(t-M))=$$
$$\xi_{c,n}(t)+\beta_{c,n}(t)+\xi_{g,n}(t)+\beta_{g,n}(t)+\left(g_n(\boldsymbol{X}(t-M))-g_n(\boldsymbol{X}(t))\right)+g_n(\boldsymbol{X}(t))\triangleq$$
$$\xi_{c,n}(t)+\beta_{c,n}(t)+\xi_{g,n}(t)+\beta_{g,n}(t)+\beta_{d,n}(t)+g_n(\boldsymbol{X}(t)) \quad (48)$$

where, similarly to how we bounded (37)

$$\|\beta_{d,n}(t)\|=\left\|\sum_{m=1}^{N}\nabla_{\boldsymbol{x_n}}u_m(\boldsymbol{X}(t-M))-\sum_{m=1}^{N}\nabla_{\boldsymbol{x_n}}u_m(\boldsymbol{X}(t))\right\|\leq$$
$$NL\sum_{i=0}^{M-1}\|\boldsymbol{X}(t-i)-\boldsymbol{X}(t-i-1)\|\leq$$
$$2NL\sum_{i=M+1}^{2M}\frac{\eta(t-i)}{\delta(t-i)}\sum_{n=1}^{N}\left\|\boldsymbol{Z}_n(t-i)\sum_{m\in\mathcal{U}_n^M(t-i+M)}a_m^n(t)\,u_m(t-i)\right\|\leq$$
$$2\frac{N^2L\left(\max_n d_n\right)}{\varepsilon_0}U_{\max}\sum_{i=M+1}^{2M}\frac{\eta(t-i)}{\delta(t-i)}\leq 2\frac{N^2L\left(\max_n d_n\right)}{\varepsilon_0}U_{\max}M\frac{\eta(t-2M)}{\delta(t-2M)} \quad (49)$$

so $\beta_{d,n}(t)$ is bounded and $\beta_{d,n}(t)\to 0$ with probability 1, for all n, since $p-q>0$. Combining Lemma 2, Lemma 3 and (49), we conclude that $\beta_n(t)\triangleq\beta_{c,n}(t)+\beta_{g,n}(t)+\beta_d(t)$ is bounded and converges to zero with probability 1. Lemma 2 and Lemma 3 also show that there exists $T>0$ such that $\xi_n(t)\triangleq\xi_{c,n}(t)+\xi_{g,n}(t)$ satisfies, for every n and every $\mu>0$

$$\lim_{j_0\to\infty}\mathbb{P}\left(\sup_{j\geq j_0}\max_{0\leq\tau\leq T}\left|\sum_{t=\kappa(jT)}^{\kappa(jT+\tau)-1}\eta(t)\,\xi_n(t)\right|\geq\mu\right)=0. \quad (50)$$

Since $\eta(t)=\frac{\eta_0}{t^p}$ for $p<1$ then $\sum_{t=1}^{\infty}\eta(t)=\infty$ and $\eta(t)\to 0$ as $t\to\infty$. The function $g(\boldsymbol{x})=(g_1(\boldsymbol{x}),...,g_N(\boldsymbol{x}))$ consists of the elements $g_n(\boldsymbol{x})=\sum_{m=1}^{N}\nabla_{\boldsymbol{x_n}}u_m(\boldsymbol{x})$ where $u_m(\boldsymbol{x})$ is continuously differentiable for each m. Hence, by Theorem 2, $\boldsymbol{x}(t)$ converges to the set of KKT stationary points of $\sum_{m=1}^{N}\nabla u_m(\boldsymbol{x})$ with probability 1.

Then if $\sum_{m=1}^{N}u_m(\boldsymbol{x})$ is concave and for all n there exists an $\boldsymbol{x}_n$ such that $q_i^n(\boldsymbol{x}_n)<0$ for all i (Slater's condition), then the duality gap is zero and the set of KKT stationary points coincides with $\arg\max_{\boldsymbol{x}\in\mathcal{A}}\sum_{n=1}^{N}u_n(\boldsymbol{x})$ [58, Proposition 5.3.1, Page 512]. □

12 Stochastic Approximation - Kushner-Clark Theorem

For convenience, we summarize here the part of [52, Theorem 5.3.1, Page 191] that we use. The Theorem in [52, Theorem 5.3.1, Page 191] is more general, and is applicable for any continuous $g(\boldsymbol{x})$, not necessarily of the form $g(\boldsymbol{x})=-\nabla f(\boldsymbol{x})$ for some continuously differentiable f.

Theorem 2 (Kushner and Clark [52, Theorem 5.3.1, Page 191])**.** *Consider the algorithm*

$$\boldsymbol{X}(t+1) = \prod_C \left(\boldsymbol{X}(t) + \eta(t)\,\boldsymbol{Y}(t)\right) \tag{51}$$

where $\boldsymbol{X}(t) \in \mathbb{R}^d$, $\prod_C$ is the projection onto $C \subset \mathbb{R}^d$ and

$$\boldsymbol{Y}(t) = g(\boldsymbol{X}(t)) + \xi(t) + \beta(t) \tag{52}$$

such that the following assumptions hold:

1. *C is closed and bounded and of the form*

$$C = \left\{\boldsymbol{x} \in \mathbb{R}^d \,|\, q_i(\boldsymbol{x}) \le 0,\ i = 1, \ldots, s\right\} \tag{53}$$

for some positive integer s, where for each i, $q_i(\boldsymbol{x})$ is continuously differentiable and for each $\boldsymbol{x}$, the gradients $\nabla q_i(\boldsymbol{x})$ for all i such that $q_i(\boldsymbol{x}) = 0$ (active constraints) are linearly independent.

2. *$\sum_t \eta(t) = \infty$ and $\eta(t) \to 0$ as $t \to \infty$ and $\eta(t) > 0$ for all t.*

3. *$g(\boldsymbol{x}) = -\nabla f(\boldsymbol{x})$ for a continuously differentiable function $f : \mathbb{R}^d \to \mathbb{R}$.*

4. *$\beta(t)$ is bounded and converges to zero with probability 1.*

5. *Define $\tau(t) \triangleq \sum_{i=1}^{t-1} \eta(i)$. For any real τ, let $\kappa(\tau)$ denote the unique value of t such that $\tau(t) \le \tau \le \tau(t+1)$. There exists $T > 0$ such that for all $\mu > 0$:*

$$\lim_{j_0 \to \infty} \mathbb{P}\left(\sup_{j \ge j_0} \max_{0 \le \tau \le T} \left|\sum_{t=\kappa(jT)}^{\kappa(jT+\tau)-1} \eta(t)\,\xi(t)\right| \ge \mu\right) = 0. \tag{54}$$

Then $\boldsymbol{X}(t)$ converges with probability 1 to the set of KKT points:

$$\mathcal{KKT} = \left\{\boldsymbol{x} \,|\, \exists \boldsymbol{\lambda} \in \mathbb{R}^s \text{ s.t. } \forall i, \lambda_i \ge 0 \text{ and } \nabla f(\boldsymbol{x}) + \sum_{i: q_i(\boldsymbol{x}) = 0} \lambda_i \nabla q_i(\boldsymbol{x}) = 0\right\}. \tag{55}$$