

A Upper Bound

We provide the proof in the following three subsections. We will repeatedly use the following result.

Lemma 7. For all $u > 0$, $\sum_{t=u+1}^{\infty} \mathbb{P}(\mathcal{B}_{k^*,t} < \mu_*) \leq \frac{M\nu}{\rho-2} \frac{1}{u^{\rho-2}}$.

Proof. The proof is straightforward using a union bound.

$$\begin{aligned} \mathbb{P}(\mathcal{B}_{k^*,t} < \mu_*) &= \mathbb{P}(\exists m \in \{1, \dots, M\}, \exists 1 \leq s \leq t-1, \mathcal{B}_{k^*,t}^{(m)}(s) \leq \mu_*) \\ &= \sum_{m=1}^M \sum_{s=1}^{t-1} \mathbb{P}\left(\bar{X}_{k^*,s}^{(m)} - \mu_{k^*}^{(m)} < \mu_* - \mu_{k^*}^{(m)} - \zeta^{(m)} - \psi^{-1}\left(\frac{\rho \log(t)}{s}\right)\right) \\ &\leq \sum_{m=1}^M \sum_{s=1}^{t-1} \nu t^{-\rho} \leq M\nu t^{1-\rho} \end{aligned} \quad (8)$$

In the third step we have used $\mu_* - \mu_k^{(m)} \leq \zeta^{(m)}$. The result follows by bounding the sum with the integral $\sum_{t=u+1}^{\infty} t^{1-\rho} \leq \int_u^{\infty} t^{1-\rho} = u^{2-\rho}/(\rho-2)$. \square

A.1 Proof of Lemma 5

We first provide a formal statement of Lemma 5.

Lemma 8. Let $m \leq M$ and consider any arm $k \in \mathcal{K}^{(m)}$. After n time steps of (γ, ρ) -MF-UCB with $\rho > 2$ and $\gamma > 0$, we have the following bounds on $\mathbb{E}[T_{k,n}^{(\ell)}]$ for $\ell = 1, \dots, M$.

$$T_{k,n}^{(\ell)} \leq \frac{\rho \log(n)}{\psi(\gamma^{(m)})} + 1, \quad \forall \ell < m, \quad \mathbb{E}[T_{k,n}^{(m)}] \leq \frac{\rho \log(n)}{\psi(\Delta_k^{(m)}/2)} + \kappa_\rho, \quad \mathbb{E}[T_{k,n}^{(>m)}] \leq \kappa_\rho.$$

Here, $\kappa_\rho = 1 + \frac{\nu}{2} + \frac{M\nu}{\rho-2}$ is a constant.

Proof. As n is fixed in this proof, we will write $\mathbb{E}[\cdot], \mathbb{P}(\cdot)$ for $\mathbb{E}[\cdot|N = n], \mathbb{P}(\cdot|N = n)$. Let $\phi_t^{(m)} = \lfloor \frac{\rho \log(t)}{\psi(\gamma^{(m)})} \rfloor$. By design of the algorithm we won't play any arm more than $\phi_n^{(m)} + 1$ times at any $m < M$. To see this, assume we have already played $\phi_n^{(m)} + 1$ times at any $t < n$. Then,

$$\psi^{-1}\left(\frac{\rho \log(t)}{T_{k,t-1}^{(m)}}\right) < \psi^{-1}\left(\frac{\rho \log(t)}{\rho \log(n)} \psi(\gamma^{(m)})\right) \leq \gamma^{(m)},$$

and we will proceed to the $(m+1)^{\text{th}}$ fidelity in step 2 of Algorithm 1. This gives the first part of the theorem. For any $\ell \geq m$ we can avoid the $\frac{1}{\psi(\gamma^{(m)})}$ dependence to obtain tighter bounds.

For the case $\ell = m$, our analysis follows usual multi-armed bandit analyses [2, 5]. For any $u \leq n$, we can bound $T_{k,n}^{(m)}$ via $T_{k,n}^{(m)} \leq u + \sum_{t=u+1}^n Z_{k,t,u}^{(m)}$ where $Z_{k,t,u}^{(m)} = \mathbb{1}\{m_t = m \wedge I_t = k \wedge T_{k,t-1}^{(m)} \geq u\}$. We relax $Z_{k,t,u}^{(m)}$ further via,

$$\begin{aligned} Z_{k,t,u}^{(m)} &\leq \mathbb{1}\{T_{k,t-1}^{(\ell)} > \phi_t^{(\ell)} \forall \ell \leq m-1 \wedge u \leq T_{k,t-1}^{(m)} \leq \phi_t^{(m)} \wedge \mathcal{B}_{k,t} > \mathcal{B}_{k^*,t}\} \\ &\leq \mathbb{1}\{T_{k,t-1}^{(m)} \geq u \wedge \mathcal{B}_{k,t}^{(m)}(T_{k,t-1}^{(m)}) \geq \mu_*\} + \mathbb{1}\{\mathcal{B}_{k^*,t} < \mu_*\} \\ &\leq \mathbb{1}\{\exists u \leq s \leq t-1 : \mathcal{B}_{k,t}^{(m)}(s) > \mu_*\} + \mathbb{1}\{\mathcal{B}_{k^*,t} < \mu_*\}. \end{aligned}$$

This yields, $\mathbb{E}[T_{k,n}^{(m)}] \leq u + \sum_{t=u+1}^n \sum_{s=u}^{t-1} \mathbb{P}(\mathcal{B}_{k,t}^{(m)}(s) > \mu_*) + \sum_{t=u+1}^n \mathbb{P}(\mathcal{B}_{k^*,t} < \mu_*)$. The third term in this summation is bounded by $M\nu/(\rho-2)$ using Lemma 7. To bound the second, choose $u = \lceil \rho \log(n)/\psi(\Delta_k^{(m)}/2) \rceil$. Then,

$$\mathbb{P}(\mathcal{B}_{k,t}^{(m)}(s) > \mu_*) = \mathbb{P}\left(\bar{X}_{k,s}^{(m)} - \mu_k^{(m)} > \mu_* - \mu_k^{(m)} - \zeta^{(m)} - \psi^{-1}\left(\frac{\rho \log(t)}{s}\right)\right)$$

$$\leq \mathbb{P}(\bar{X}_{k,s}^{(m)} - \mu_k^{(m)} > \Delta_k^{(m)}/2) \leq \nu \exp\left(-s\psi\left(\frac{\Delta_k^{(m)}}{2}\right)\right) \leq \nu n^{-\rho} \quad (9)$$

In the second and last steps we have used $\psi^{-1}(\rho \log(t)/s) < \psi^{-1}(\rho \log(t)/u) \leq \Delta_k^{(m)}/2$ since ψ^{-1} is increasing and $u > \rho \log(n)/\psi(\Delta_k^{(m)}/2)$. Since there are at most n^2 terms in the summation, the second term is bounded by $\nu n^{2-\rho}/2 \leq \nu/2$. Collecting the terms gives the bound on $\mathbb{E}[T_{k,n}^{(m)}]$.

To bound $T_{k,n}^{(>m)}$ we write $T_{k,n}^{(>m)} \leq u + \sum_{t=u+1}^n Z_{k,t,u}^{(>m)}$ where

$$\begin{aligned} Z_{k,t,u}^{(>m)} &= \mathbb{1}\{m_t > m \quad \wedge \quad I_t = k \quad \wedge \quad T_{k,t-1}^{(>m)} \geq u\} \\ &\leq \mathbb{1}\{T_{k,t-1}^{(\ell)} > \phi_t^{(\ell)} \forall \ell \leq m \quad \wedge \quad \mathcal{B}_{k,t} > \mathcal{B}_{k^*,t} \quad \wedge \quad T_{k,t-1}^{(>m)} \geq u\} \\ &\leq \mathbb{1}\{T_{k,t-1}^{(m)} > \phi_t^{(m)} \quad \wedge \quad \mathcal{B}_{k,t}^{(m)}(T_{k,t-1}^{(m)}) > \mu_\star\} + \mathbb{1}\{\mathcal{B}_{k^*,t} < \mu_\star\} \\ &\leq \mathbb{1}\{\exists \phi_t^{(m)} + 1 \leq s \leq t-1 : \mathcal{B}_{k,t}^{(m)}(s) > \mu_\star\} + \mathbb{1}\{\mathcal{B}_{k^*,t} < \mu_\star\} \end{aligned}$$

This yields, $\mathbb{E}[T_{k,n}^{(>m)}] \leq u + \sum_{t=u+1}^n \sum_{s=\phi_t^{(m)}+1}^{t-1} \mathbb{P}(\mathcal{B}_{k,t}^{(m)}(s) > \mu_\star) + \sum_{t=u+1}^n \mathbb{P}(\mathcal{B}_{k^*,t} < \mu_\star)$. The inner term inside the double summation can be bounded via,

$$\begin{aligned} \mathbb{P}(\mathcal{B}_{k,t}^{(m)}(s) > \mu_\star) &= \mathbb{P}\left(\bar{X}_{k,s}^{(m)} - \mu_k^{(m)} > \mu_\star - \mu_k^{(m)} - \zeta^{(m)} - \psi^{-1}\left(\frac{\rho \log(t)}{s}\right)\right) \\ &\leq \mathbb{P}(\bar{X}_{k,s}^{(m)} - \mu_k^{(m)} > \Delta_k^{(m)} - \gamma^{(m)}) \leq \nu \exp(-s\psi(\Delta_k^{(m)} - \gamma^{(m)})) \\ &\leq \nu \exp\left(-\frac{\psi(\Delta_k^{(m)} - \gamma^{(m)})}{\psi(\gamma^{(m)})} \rho \log(t)\right) \leq \nu t^{-\rho} \end{aligned} \quad (10)$$

The second step follows from $s > \phi_t^{(m)} > \rho \log(t)/\psi(\gamma^{(m)})$ and the last step uses $\psi(\Delta_k^{(m)} - \gamma^{(m)}) > \psi(\gamma^{(m)})$ when $\Delta_k^{(m)} > 2\gamma^{(m)}$. To bound the summation, we use $u = 1$ and bound it by an integral: $\sum_{t=u+1}^n t^{-\rho+1} \leq 1/(2u^{\rho-2}) \leq 1/2$. Collecting the terms gives the bound on $\mathbb{E}[T_{k,n}^{(>m)}]$. \square

A.2 Proof of Lemma 6

We first provide a formal statement of Lemma 6.

Lemma 9. Consider any arm $k \in \mathcal{K}^{(m)}$. For (γ, ρ) -MF-UCB with $\rho > 2$ and $\gamma > 0$, we have the following concentration results for $\ell = 1, \dots, M$ for any $x \geq 1$.

$$\begin{aligned} \mathbb{P}\left(T_{k,n}^{(m)} > x \left(1 + \frac{\rho \log(n)}{\psi(\Delta_k^{(m)}/2)}\right)\right) &\leq \frac{\nu \tilde{\kappa}_{k,\rho}^{(m)}}{(x \cdot \log(n))^{\rho-1}} + \frac{\nu}{n^{x\rho-1}}. \\ \mathbb{P}\left(T_{k,n}^{(>m)} > x\right) &\leq \frac{M\nu}{\rho-1} \frac{1}{x^{\rho-1}} + \frac{1}{(\rho-2)x^{\rho-2}} \end{aligned}$$

$$\text{Here, } \tilde{\kappa}_{k,\rho}^{(m)} = \frac{M}{\rho-1} \left(\frac{\psi(\Delta_k^{(m)}/2)}{\rho}\right)^{\rho-1}.$$

Proof. For the first inequality, we modify the analysis in Audibert et al. [2] to the multi-fidelity setting. We begin with the following observation for all $u \in \mathbb{N}$.

$$\begin{aligned} \{\forall t : u+1 \leq t \leq n, \mathcal{B}_{k,t}^{(m)}(u) \leq \mu_\star\} &\cap \\ \bigcap_{m=1}^M \{\forall 1 \leq s \leq n-u : \mathcal{B}_{k^*,u+s}^{(m)}(s) > \mu_\star\} &\implies T_{k,n}^{(m)} \leq u \end{aligned} \quad (11)$$

To prove this, consider $s^{(m)}, m = 1, \dots, M$ such that $s^{(1)} \geq 1, s^{(m)} \geq 0, \forall m \neq 1$. For all $u + \sum_{m=1}^M s^{(m)} \leq t \leq n$ and for all $\ell = 1, \dots, M$ we have

$$\mathcal{B}_{k^*,t}^{(\ell)}(s^{(\ell)}) \geq \mathcal{B}_{k^*,u+s}^{(\ell)}(s^{(\ell)}) > \mu_\star \geq \mathcal{B}_{k,t}^{(m)}(u) \geq \mathcal{B}_{k,t}^{(m)}(T_{k,t-1}^{(m)}).$$

This means that arm k will not be the $\mathcal{B}_{k,t}$ maximiser at any time $u < t < n$ and consequently it won't be played more than $u + 1$ times at the m^{th} fidelity. Via the union bound we have,

$$\mathbb{P}(T_{k,n}^{(m)} > u) \leq \sum_{t=u+1}^n \mathbb{P}(\mathcal{B}_{k,t}^{(m)}(u) > \mu_\star) + \sum_{m=1}^M \sum_{s=1}^{n-u} \mathbb{P}(\mathcal{B}_{k_\star, u+s}^{(m)}(s) < \mu_\star).$$

We will use $u = \lceil x(1 + \rho \log(n)/\psi(\Delta_k^{(m)}/2)) \rceil$. Bounding the inner term of the second double summation closely mimics the calculations in (8) via which it can be shown $\mathbb{P}(\mathcal{B}_{k_\star, u+s}^{(m)}(s) < \mu_\star) \leq \nu(u+s)^{-\rho}$. The second term is then bounded by an integral as follows,

$$\sum_{m=1}^M \sum_{s=1}^{n-u} \mathbb{P}(\mathcal{B}_{k_\star, u+s}^{(m)}(s) < \mu_\star) \leq M \sum_{s=1}^{n-u} \nu(u+s)^{-\rho} \leq M\nu \int_u^n t^{-\rho} \leq \frac{M\nu u^{1-\rho}}{\rho-1} \leq \frac{\nu \tilde{\kappa}_{k,\rho}^{(m)}}{(x \cdot \log(n))^{\rho-1}}$$

The inner term of the first summation mimics the calculations in (9). Noting that $s > x\rho \log(n)/\psi(\Delta_k^{(m)}/2)$ it can be shown $\mathbb{P}(\mathcal{B}_{k,t}^{(m)}(u) > \mu_\star) \leq \nu n^{-\rho x}$ which bounds the outer summation by $\nu n^{-\rho x+1}$. This proves the first concentration result.

For the second, we begin with the following observation for all $u \in \mathbb{N}$.

$$\{\forall t : u+1 \leq t \leq n, \mathcal{B}_{k,t}^{(m)}(T_{k,t-1}^{(m)}) \leq \mu_\star \quad \vee \quad T_{k,t-1}^{(m)} \leq \phi_t^{(m)}\} \cap \bigcap_{m=1}^M \{\forall 1 \leq s \leq n-u : \mathcal{B}_{k_\star, u+s}^{(m)}(s) > \mu_\star\} \implies T_{k,n}^{(>m)} \leq u \quad (12)$$

To prove this first note that when $T_{k,t-1}^{(m)} \leq \phi_t^{(m)}$ we will play at the m^{th} fidelity or lower. Otherwise, consider $s^{(m)}, m = 1, \dots, M$ such that $s^{(1)} \geq 1$ and $s^{(m)} \geq 0, \forall m$. For all $u + \sum_{m=1}^M s^{(m)} \leq t \leq n$ and for all $\ell = 1, \dots, M$ we have

$$\mathcal{B}_{k_\star, t}^{(\ell)}(s^{(\ell)}) \geq \mathcal{B}_{k_\star, u+s}^{(\ell)}(s^{(\ell)}) > \mu_\star \geq \mathcal{B}_{k,t}^{(m)}(T_{k,t-1}^{(m)}).$$

This means that arm k will not be played at time t and consequently for any $t > u$. After a further relaxation we get,

$$\mathbb{P}(T_{k,n}^{(>m)} > u) \leq \sum_{t=u+1}^n \sum_{s=\phi_t^{(m)}+1}^{t-1} \mathbb{P}(\mathcal{B}_{k,t}^{(m)}(s) > \mu_\star) + \sum_{m=1}^M \sum_{s=1}^{n-u} \mathbb{P}(\mathcal{B}_{k_\star, u+s}^{(m)}(s) < \mu_\star)$$

The second summation is bounded via $\frac{M\nu}{(\rho-1)u^{\rho-1}}$. Following an analysis similar to (10), the inner term of the first summation can be bounded by $\nu t^{-\rho}$ which bounds the first term by $u^{2-\rho}/(\rho-2)$. The result follows by using $u = x$ in (12). \square

A.3 Proof of Theorem 2

We first establish the following Lemma.

Lemma 10 (Regret of MF-UCB). *Let $\rho > 4$. There exists Λ_0 depending on $\lambda^{(1)}, \lambda^{(M)}$ such that for all $\Lambda > \Lambda_0$, (γ, ρ) -MF-UCB satisfies,*

$$\mathbb{E}[R(\Lambda)] \leq \mu_\star \lambda^{(M)} + \sum_{k=1}^K \Delta_k^{(M)} \left(\sum_{\ell=1}^{\lfloor k \rfloor - 1} \lambda^{(\ell)} \frac{\rho(\log(n_\Lambda) + c)}{\psi(\gamma^{(\ell)})} + \lambda^{(\lfloor k \rfloor)} \frac{\rho(\log(n_\Lambda) + c)}{\psi(\Delta_k^{(\lfloor k \rfloor)}/2)} + \mu_\star \kappa_\rho \lambda^{(M)} \right)$$

Here $c = 1 + \log(2)$ and $\kappa_\rho = 1 + \frac{\nu}{\rho-2} + \frac{M\nu}{\rho-2}$ are constants.

Denote the set of arms ‘‘above’’ $\mathcal{K}^{(m)}$ by $\widehat{\mathcal{K}}^{(m)} = \bigcup_{\ell=m+1}^M \mathcal{K}^{(\ell)}$ and those ‘‘below’’ $\mathcal{K}^{(m)}$ by $\check{\mathcal{K}}^{(m)} = \bigcup_{\ell=1}^{m-1} \mathcal{K}^{(\ell)}$. We first observe,

$$\left(\forall m \leq M-1, \forall k \in \mathcal{K}^{(m)}, T_{k,n}^{(m)} \leq x \left(1 + \frac{\rho \log(n)}{\psi(\Delta_k^{(m)}/2)} \right) \quad \wedge \quad T_{k,n}^{(>m)} \leq y \right) \quad (13)$$

$$\implies \sum_{m=1}^{M-1} Q_n^{(m)} \leq Ky + \sum_{m=1}^{M-1} \sum_{k \in \mathcal{K}^{(m)}} x \left(1 + \frac{\rho \log(n)}{\psi(\Delta_k^{(m)}/2)} \right) + \sum_{m=1}^{M-1} |\widehat{\mathcal{K}}^{(m)}| \left(1 + \frac{\rho \log(n)}{\psi(\gamma^{(m)})} \right)$$

To prove this we first note that the LHS of (13) is reducible to,

$$\forall m \leq M-1, \quad Q_n^{(m)} \leq \sum_{k \in \tilde{\mathcal{K}}^{(m)}} T_{k,n}^{(m)} + \sum_{k \in \mathcal{K}^{(m)}} x \left(1 + \frac{\rho \log(n)}{\psi(\Delta_k^{(m)}/2)} \right) + \sum_{k \in \tilde{\mathcal{K}}^{(m)}} \left(1 + \frac{\rho \log(n)}{\psi(\gamma^{(m)})} \right)$$

The statement follows by summing the above from $m = 1, \dots, M-1$ and rearranging the $T_{k,n}^{(>m)}$ terms to obtain,

$$\begin{aligned} \sum_{m=1}^{M-1} \sum_{k \in \tilde{\mathcal{K}}^{(m)}} T_{k,n}^{(m)} &= \sum_{m=1}^{M-1} \sum_{\ell=1}^{m-1} \sum_{k \in \mathcal{K}^{(\ell)}} T_{k,n}^{(m)} = \sum_{m=1}^{M-2} \sum_{k \in \mathcal{K}^{(m)}} \sum_{\ell=m+1}^{M-1} T_{k,n}^{(\ell)} \leq \sum_{m=1}^{M-2} \sum_{k \in \mathcal{K}^{(m)}} T_{k,n}^{(>m)} \\ &\leq (K - |\mathcal{K}^{(M-1)} \cup \mathcal{K}^{(M)} \cup \mathcal{K}_*|)y \leq Ky. \end{aligned}$$

Now for the given Λ under consideration, define $\delta_\Lambda = \frac{1}{\log(\Lambda/\lambda^{(1)})}$. In addition define,

$$\begin{aligned} x_{n,\delta} &= \max \left(1, \frac{1}{\rho} \left(3 + \frac{\log(2\nu\pi^2 K/(3\delta))}{\log(n)} \right), \left(\frac{2\pi^2 K\nu M}{3(\rho-1)\delta} \right)^{\frac{1}{\rho-1}} \frac{\psi(\Delta_k^{(m)}/2)}{\rho} n^{\frac{2}{\rho-1}} \right). \\ y_{n,\delta} &= \max \left(1, \left(\frac{2\pi^2 K M \nu}{3(\rho-1)\delta} \right)^{\frac{1}{\rho-1}} n^{\frac{2}{\rho-1}}, \left(\frac{\pi^2 K}{3\delta} \right)^{\frac{1}{\rho-2}} n^{\frac{2}{\rho-2}} \right). \end{aligned}$$

Now choose $n_{0,\Lambda}$ to be the smallest n such that the following holds for all $n \geq n_{0,\Lambda}$.

$$\frac{n}{2} \geq Ky_{n,\delta_\Lambda} + \sum_{m=1}^{M-1} \sum_{k \in \mathcal{K}^{(m)}} x_{n,\delta_\Lambda} \left(1 + \frac{\rho \log(n)}{\psi(\Delta_k^{(m)}/2)} \right) + \sum_{m=1}^{M-1} \sum_{k \in \tilde{\mathcal{K}}^{(m)}} 1 + \frac{\rho \log(n)}{\psi(\gamma)}, \quad (14)$$

For such an $n_{0,\Lambda}$ to exist, for a given Λ , we need both x_n, y_n sublinear. This is true since $\rho > 4$. In addition, observe that $n_{0,\Lambda}$ grows only polylogarithmically in Λ since (14) reduces to $n^p \gtrsim (\log(\Lambda))^{1/2}$ where $p > 0$ depends on our choice of ρ .

By (13), the RHS of (14) is an upper bound on the number of plays at fidelities lower than M . Therefore, for all $n \geq n_{0,\Lambda}$,

$$\begin{aligned} \mathbb{P}\left(Q_n^{(M)} < \frac{n}{2}\right) &\leq \sum_{m=1}^{M-1} \sum_{k \in \mathcal{K}^{(m)}} \mathbb{P}\left(T_{k,n}^{(m)} > x_{n,\delta} \left(1 + \frac{\rho \log(n)}{\psi(\Delta_k^{(m)}/2)} \right)\right) + \mathbb{P}\left(T_{k,n}^{(>m)} > y_{n,\delta}\right) \quad (15) \\ &\leq \sum_{m=1}^{M-1} \sum_{k \in \mathcal{K}^{(m)}} \frac{\nu}{n^{\rho x_{n,\delta_\Lambda}-1}} + \frac{\nu \tilde{\kappa}_{k,\rho}^{(m)}}{(x_{n,\delta_\Lambda} \log(n))^{\rho-1}} + \frac{\nu M}{(\rho-1)y_{n,\delta_\Lambda}^{\rho-1}} + \frac{1}{2y_{n,\delta_\Lambda}^{\rho-2}} \\ &\leq K \left(4 \times \frac{3\delta}{2Kn^2\pi^2} \right) \leq \frac{6\delta}{n^2\pi^2}. \end{aligned}$$

The last step follows from the fact that each of the four terms inside the summation in the second line are $\leq 3\delta/(2Kn^2\pi^2)$. For the last term we have used that $(\rho-2)/2 > 1$ and that $3\delta/(\pi^2 K)$ is smaller than 1. Note that the double summation just enumerates over all arms in \mathcal{K} .

We can now specify the conditions on Λ_0 . Λ_0 should be large enough so that for all $\Lambda \geq \Lambda_0$, we have $\lfloor \Lambda/\lambda^{(M)} \rfloor \geq n_{0,\Lambda}$. Such an Λ_0 exists since $n_{0,\Lambda}$ grows only polylogarithmically in Λ . This ensures that we have played a sufficient number of rounds to apply the concentration result in (15).

Let the (random) expended capital after n rounds of MF-UCB be $\Omega(n)$. Let $\mathcal{E} = \{\exists n \geq n_{0,\Lambda} : \Omega(n) < n\lambda^{(M)}/2\}$. Since $\Omega(n) \geq \lambda^{(M)}Q_n^{(M)}$, by using the union bound on (15) we have $\mathbb{P}(\mathcal{E}) \leq \delta_\Lambda$. Therefore,

$$\mathbb{P}\left(N > \frac{2\Lambda}{\lambda^{(M)}}\right) = \mathbb{P}\left(N > \frac{2\Lambda}{\lambda^{(M)}} \middle| \mathcal{E}\right) \underbrace{\mathbb{P}(\mathcal{E})}_{\leq \delta_\Lambda} + \underbrace{\mathbb{P}\left(N > \frac{2\Lambda}{\lambda^{(M)}} \middle| \mathcal{E}^c\right)}_{=0} \mathbb{P}(\mathcal{E}^c) < \delta_\Lambda$$

The last step uses the following reasoning: Conditioned on \mathcal{E}^c , $n > 2\Omega(n)/\lambda^{(M)}$ is false for $n > n_{0,\Lambda}$. In particular, it is true for the *random* number of plays N since $\Lambda > \Lambda_0 \implies N \geq n_{0,\Lambda}$. Now, clearly $\Lambda > \Omega(N)$ and therefore $N > 2\Lambda/\lambda^{(M)}$ is also false.

By noting that $n_\Lambda = \Lambda/\lambda^{(M)}$ and that $\log(\Lambda/\lambda^{(1)})$ is always an upper bound on $\log(N)$, we have,

$$\mathbb{E}[\log(N)] \leq \log(2n_\Lambda)\mathbb{P}(N < 2n_\Lambda) + \log\left(\frac{\Lambda}{\lambda^{(1)}}\right)\mathbb{P}(N > 2n_\Lambda) \leq \log(n_\Lambda) + 1 + \log(2) \quad (16)$$

Lemma 10 now follows by an application of Lemma 5. First we condition on $N = n$ to obtain,

$$\begin{aligned} \mathbb{E}[R(\Lambda)|N = n] &\leq \mu_\star \lambda^{(M)} + \sum_{k=1}^K \sum_{m=1}^M \Delta_k^{(M)} \lambda^{(m)} T_{k,n}^{(m)} \\ &\leq \mu_\star \lambda^{(M)} + \sum_{k=1}^K \Delta_k^{(M)} \left(\sum_{\ell=1}^{[k]-1} \lambda^{(\ell)} \frac{\rho \log(n)}{\psi(\gamma^{(m)})} + \lambda^{([k])} \frac{\rho \log(n)}{\psi(\Delta_k^{([k])}/2)} + \kappa_\rho \lambda^{(M)} \right) \end{aligned}$$

The theorem follows by plugging in the above in $\mathbb{E}[R(\Lambda)] = \mathbb{E}[\mathbb{E}[R(\Lambda)|N]]$ and using the bound for $\mathbb{E}[\log(N)]$ in (16).

We can now bound the regret for MF-UCB.

Proof of Theorem 2. Recall that $\psi(\gamma^{(m)}) = \frac{\lambda^{(m)}}{\lambda^{(m+1)}} \psi(\zeta^{(m)})$. Plugging this into Lemma 10 we get

$$\begin{aligned} \mathbb{E}[R(\Lambda)] &\leq \mu_\star \lambda^{(M)} + \sum_{k=1}^K \Delta_k^{(M)} \left(\sum_{\ell=1}^{[k]-1} \lambda^{(\ell+1)} \frac{\rho(\log(n_\Lambda) + c)}{\psi(\zeta^{(\ell)})} + \lambda^{([k])} \frac{\rho(\log(n_\Lambda) + c)}{\psi(\Delta_k^{([k])}/2)} + \kappa_\rho \lambda^{(M)} \right) \\ &\leq \mu_\star \lambda^{(M)} + \sum_{k=1}^K \Delta_k^{(M)} \cdot \lambda^{([k])} \rho(\log(n_\Lambda) + c) \left(\frac{2}{\psi(\zeta^{([k]-1)})} + \frac{1}{\psi(\Delta_k^{([k])}/2)} \right) + \Delta_k^{(M)} \kappa_\rho \lambda^{(M)} \end{aligned}$$

The second step uses Assumption 1. The theorem follows by noting that for any $k \in \mathcal{K}^{(m)}$ and $\ell < m$, $\Delta_k^{(m)} = \Delta_k^{(\ell)} + \zeta^{(\ell)} - \zeta^{(m)} + \mu_k^{(\ell)} - \mu_k^{(M)} + \mu_k^{(M)} - \mu_k^{(m)} \leq 2\gamma^{(\ell)} + 2\zeta^{(\ell)} \leq 4\zeta^{(\ell)}$. Therefore $1/\psi(\Delta_k^{(m)}) > c_1/\psi(\zeta^{(\ell)})$ where c_1 depends on ψ (for sub-Gaussian distributions, $c_1 = 1/16$). \square

B Lower Bound

The regret R_k incurred by any multi-fidelity strategy after capital Λ due to a suboptimal arm k is,

$$R_k(\Lambda) = \Delta_k^{(M)} \sum_{m=1}^M \lambda^{(m)} T_{k,N}^{(m)},$$

here N is the total number of plays. We then have, $R(\Lambda) = \sum_k R_k(\Lambda)$. For what follows, for an arm k and any fidelity m denote $\text{KL}_k^{(m)} = \text{KL}(\mu_k^{(m)} \parallel \mu_\star - \zeta^{(m)})$. The following lemma provides an asymptotic lower bound on R_k .

Lemma 11. Consider any set of Bernoulli reward distributions with $\mu_\star \in (1/2, 1)$ and $\zeta^{(1)} < 1/2$. For any k with $\Delta_k^{(\ell)} < 0$ for all $\ell < p$ and $\Delta_k^{(p)} > 0$, there exists a problem dependent constant c_p such that any strategy satisfying Assumption 3 must satisfy,

$$\liminf_{\Lambda \rightarrow \infty} \frac{R_k(\Lambda)}{\log(n_\Lambda)} \geq c'_p \Delta_k^{(M)} \min_{\ell \geq p, \Delta_k^{(\ell)} > 0} \frac{\lambda^{(\ell)}}{\Delta_k^{(\ell)^2}}$$

Proof. For now we will fix $N = n$ and consider any game after n rounds. Our proof we will modify the reward distributions of the given arm k for all $\ell \geq p$ and show that any algorithm satisfying Assumption 3 will not be able to distinguish between both problems with high probability. Since

the KL divergence is continuous, for any $\epsilon > 0$ we can choose $\tilde{\mu}_k^{(p)} \in (\mu_\star - \zeta^{(p)}, \mu_\star - \zeta^{(p)} + \min_{\ell < p} -\Delta_k^{(\ell)})$ such that $\text{KL}(\mu_k^{(p)} \parallel \tilde{\mu}_k^{(p)}) < (1 + \epsilon)\text{KL}(\mu_k^{(p)} \parallel \mu_\star - \zeta^{(p)}) = (1 + \epsilon)\text{KL}_k^{(p)}$.

The modified construction for arm k , will also have Bernoulli distributions with means $\tilde{\mu}_k^{(1)}, \tilde{\mu}_k^{(2)}, \dots, \tilde{\mu}_k^{(M)}$. $\tilde{\mu}_k^{(p)}$ will be picked to satisfy the two constraints above and for the remaining fidelities,

$$\tilde{\mu}_k^{(\ell)} = \mu_k^{(\ell)} \quad \text{for } \ell < p, \quad \tilde{\mu}_k^{(\ell)} = \tilde{\mu}_k^{(p)} + \zeta^{(p)} - \zeta^{(\ell)} \quad \text{for } \ell > p.$$

Now note that for $\ell < p$, $\tilde{\mu}_k^{(M)} - \tilde{\mu}_k^{(\ell)} = \tilde{\mu}_k^{(p)} + \zeta^{(p)} - \mu_k^{(\ell)} < \mu_\star - \zeta^{(p)} - \Delta_k^{(\ell)} + \zeta^{(p)} - \mu_k^{(\ell)} = \zeta^{(\ell)}$; similarly, $\tilde{\mu}_k^{(M)} - \tilde{\mu}_k^{(\ell)} = \tilde{\mu}_k^{(p)} + \zeta^{(p)} - \mu_k^{(\ell)} > \mu_\star - \mu_k^{(\ell)} > \mu_k^{(M)} - \mu_k^{(\ell)} > -\zeta^{(\ell)}$. For $\ell > p$, $\tilde{\mu}_k^{(M)} - \tilde{\mu}_k^{(\ell)} = \zeta^{(\ell)}$. Hence, the modified construction satisfies the conditions on the lower fidelities laid out in Section 2 and we can use Assumption 3. Further $\tilde{\mu}_k^{(M)} > \mu_\star$, so k is the optimal arm in the modified problem. Now we use a change of measure argument.

Following Bubeck and Cesa-Bianchi [5], Lai and Robbins [9], denote the expectations, probabilities and distribution in the original problem as $\mathbb{E}, \mathbb{P}, P$ and in the modified problem as $\tilde{\mathbb{E}}, \tilde{\mathbb{P}}, \tilde{P}$. Denote a sequence of observations when playing arm k at by $\{Z_{k,t}^{(\ell)}\}_{t \geq 0}$ and define,

$$L_k^{(\ell)}(s) = \sum_{t=1}^s \log \left(\frac{\mu_k^{(\ell)} Z_{k,t}^{(\ell)} + (1 - \mu_k^{(\ell)})(1 - Z_{k,t}^{(\ell)})}{\tilde{\mu}_k^{(\ell)} Z_{k,t}^{(\ell)} + (1 - \tilde{\mu}_k^{(\ell)})(1 - Z_{k,t}^{(\ell)})} \right) = \sum_{t: Z_{k,t}^{(\ell)}=1} \log \frac{\mu_k^{(\ell)}}{\tilde{\mu}_k^{(\ell)}} + \sum_{t: Z_{k,t}^{(\ell)}=0} \log \frac{1 - \mu_k^{(\ell)}}{1 - \tilde{\mu}_k^{(\ell)}}.$$

Observe that $\mathbb{E}[s^{-1} L_k^{(\ell)}(s)] = \text{KL}(\mu_k^{(\ell)} \parallel \tilde{\mu}_k^{(\ell)})$. Let A be any event in the σ -field generated by the observations in the game.

$$\begin{aligned} \tilde{\mathbb{P}}(A) &= \int \mathbb{1}(A) d\tilde{P} = \int \mathbb{1}(A) \prod_{\ell=p}^M \left(\prod_{i=1}^{T_{k,n}^{(\ell)}} \frac{\tilde{\theta}_k^{(\ell)}(Z_{k,i}^{(\ell)})}{\theta_k^{(\ell)}(Z_{k,i}^{(\ell)})} \right) dP \\ &= \mathbb{E} \left[\mathbb{1}(A) \exp \left(- \sum_{\ell \geq p} L_k^{(\ell)}(T_{k,n}^{(\ell)}) \right) \right] \end{aligned} \quad (17)$$

Now let $f_n^{(\ell)} = C \log(n)$ for all ℓ such that $\Delta_k^{(\ell)} < 0$ and $f_n^{(\ell)} = \frac{1}{M-p} \frac{1-\epsilon}{\text{KL}(\mu_k^{(p)} \parallel \tilde{\mu}_k^{(p)})} \log(n)$ otherwise.

(Recall that $\Delta_k^{(\ell)} < 0$ for all $\ell < p$ and $\Delta_k^{(p)} > 0$). C is a large enough constant that we will specify shortly. Define the following event A_n .

$$A_n = \left\{ T_{k,n}^{(\ell)} \leq f_n^{(\ell)}, \forall \ell \quad \wedge \quad L_k^{(\ell)}(T_{k,n}^{(\ell)}) \leq \frac{1}{M-p} (1 - \epsilon/2) \log(n), \forall \ell : \Delta_k^{(\ell)} > 0 \right\}$$

By (17) we have $\tilde{\mathbb{P}}(A_n) \geq \mathbb{P}(A_n) n^{-(1-\epsilon/2)}$. Since k is the unique optimal arm in the modified construction, by Assumptions 3 we have $\forall a > 0$,

$$\tilde{\mathbb{P}}(A_n) \leq \mathbb{P} \left(\sum_m T_{k,n}^{(m)} < \Theta(\log(n)) \right) \leq \frac{\mathbb{E}[n - \sum_m T_{k,n}^{(m)}]}{n - \Theta(\log(n))} \in o(n^{a-1})$$

By choosing $a < \epsilon/2$ we have $\mathbb{P}(A_n) \rightarrow 0$ as $n \rightarrow \infty$. Next, we upper bound the probability of A_n in the original problem as follows,

$$\mathbb{P}(A_n) \geq \mathbb{P} \left(\underbrace{T_{k,n}^{(\ell)} \leq f_n^{(\ell)}, \forall \ell}_{A_{n,1}} \quad \wedge \quad \underbrace{\max_{s \leq f_n} L_k^{(\ell)}(s) \leq \frac{1}{M-p} (1 - \epsilon/2) \log n, \forall \ell : \Delta_k^{(\ell)} > 0}_{A_{n,2}} \right)$$

We will now show that $A_{n,2}$ remains large as $n \rightarrow \infty$. Writing $A_{n,2} = \bigcap_{\ell: \Delta_k^{(\ell)} > 0} A_{n,2,\ell}$, we have

$$\mathbb{P}(A_{n,2,\ell}) = \mathbb{P} \left(\frac{f_n^{(\ell)}(M-p)}{(1-\epsilon)\log(n)} \cdot \frac{1}{f_n^{(\ell)}} \max_{s \leq f_n^{(\ell)}} L_k^{(m)}(s) \leq \frac{1-\epsilon/2}{1-\epsilon} \right).$$

As $f_n^{(\ell)} \rightarrow \infty$, by the strong law of large numbers $\frac{1}{f_n^{(\ell)}} \max_{s \leq f_n^{(\ell)}} L_k^{(m)}(s) \rightarrow \text{KL}(\mu_k^{(m)} \parallel \tilde{\mu}_k^{(m)})$. After substituting for $f_n^{(\ell)}$ and repeating for all ℓ , we get $\lim_{n \rightarrow \infty} \mathbb{P}(A_{n,2}) = 1$. Therefore, $\mathbb{P}(A_{n,1}) \leq o(1)$. To conclude the proof, we upper bound $\mathbb{E}[R_k(\Lambda)]$ as follows,

$$\begin{aligned} \frac{\mathbb{E}[R_k(\Lambda)]}{\Delta_k^{(M)}} &\geq \mathbb{P}(\exists \ell \text{ s.t. } T_{k,N}^{(\ell)} > f_N^{(\ell)}) \cdot \mathbb{E}[R_k(\Lambda) \mid \exists \ell \text{ s.t. } T_{k,N}^{(\ell)} > f_N^{(\ell)}] \geq \mathbb{P}(\overline{A_{n,1}}) \cdot \min_{\ell} f_{n_\Lambda}^{(\ell)} \lambda^{(\ell)} \\ &\geq (1 - o(1)) \min_{\ell \geq p} \frac{(1 - \epsilon) \log(n_\Lambda) \lambda^{(\ell)}}{(M - p) \text{KL}(\mu_k^{(\ell)} \parallel \tilde{\mu}_k^{(\ell)})} \geq \frac{\log(n_\Lambda)}{M - p} (1 - o(1)) \frac{1 - \epsilon}{1 + \epsilon} \min_{\ell > m} \frac{\lambda^{(\ell)}}{\text{KL}_k^{(\ell)}} \end{aligned}$$

Above, the second step uses the fact that $N \geq n_\Lambda$ and \log is increasing. In the third step, we have chosen $C > \max_{\ell \geq p} \lambda^{(\ell)} \Delta_k^{(M)} / \text{KL}(\mu_k^{(\ell)} \parallel \tilde{\mu}_k^{(\ell)})$ for $\ell < p$ large enough so that the minimiser will be at $\ell \geq p$. The lemma follows by noting that the statements holds for all $\epsilon > 0$ and that for Bernoulli distributions with parameters μ_1, μ_2 , $\text{KL}(\mu_1 \parallel \mu_2) \leq (\mu_1 - \mu_2)^2 / (\mu_2(1 - \mu_2))$. The constant given in the theorem is $c'_p = \frac{1}{M-p} \min_{\ell > p} (\mu_\star - \zeta^{(\ell)})(1 - \mu_\star + \zeta^{(\ell)})$. \square

We can now use the above Lemma to prove theorem 4.

Proof of Theorem 4. Let $k \in \mathcal{K}_\star^{(m)}$. We will use Lemma 11 with $p = m$. It is sufficient to show that $\lambda^{(\ell)} / \Delta_k^{(\ell)^2} \gtrsim \lambda^{(m)} / \Delta_k^{(m)^2}$ for all $\ell > m$. First note that

$$\Delta_k^{(\ell)} = \mu_\star - \mu_k^{(m)} - \zeta^{(m)} + \mu_k^{(m)} - \mu_\star + \mu_\star - \mu_k^{(\ell)} + \zeta^{(m)} - \zeta^{(\ell)} \leq \Delta_k^{(m)} + 2\zeta^{(m)} \leq 2\Delta_k^{(m)} \sqrt{\frac{\lambda^{(m+1)}}{\lambda^{(m)}}}$$

Here the last step uses that $\Delta_k^{(m)} > 2\gamma^{(m)} = \sqrt{\lambda^{(m)} / \lambda^{(m+1)}} \zeta^{(m)}$. Here we have used $\psi(\epsilon) = 2\epsilon^2$ which is just Hoeffding's inequality. Therefore, $\frac{\lambda^{(m)}}{\Delta_k^{(m)^2} \leq 4 \frac{\lambda^{(m+1)}}{\Delta_k^{(\ell)^2}} \leq 4 \frac{\lambda^{(\ell)}}{\Delta_k^{(\ell)^2}}$.

When $k \in \mathcal{K}_\star^{(m)}$, we use Lemma 11 with $p = \ell_0 = \min\{\ell; \Delta_k^{(\ell)} > 0\}$. However, by repeating the same argument as above, we can eliminate all $\ell > m$. Hence, we only need to consider ℓ such that $\ell_0 \leq \ell \leq m$ and $\Delta_k^{(\ell)} > 0$ in the minimisation of Lemma 11. This is precisely the set $\mathcal{L}_m(k)$ given in the theorem. The theorem follows by repeating the above argument for all arms $k \in \mathcal{K}$. The constant c_p in Theorem 4 is $c'_p/4$ where c'_p is from Lemma 11. \square

C Details on the Simulations

We present the details on the simulations used in the experiment. Denote $\vec{\zeta} = (\zeta^{(1)}, \zeta^{(2)}, \dots, \zeta^{(M)})$ and $\vec{\lambda} = (\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(M)})$. Figure 3 illustrates the mean values of these arms.

1. Gaussian: $M = 500$, $M = 3$, $\vec{\zeta} = (0.2, 0.1, 0)$, $\vec{\lambda} = (1, 10, 1000)$.
The high fidelity means were chosen to be a uniform grid in $(0, 1)$. The Gaussian distributions had standard deviation 0.2.
2. Gaussian: $M = 500$, $M = 4$, $\vec{\zeta} = (1, 0.5, 0.2, 0)$, $\vec{\lambda} = (1, 5, 20, 50)$.
The high fidelity means were sampled from a $\mathcal{N}(0, 1)$ distribution. The Gaussian distributions had standard deviation 1.
3. Bernoulli: $M = 200$, $M = 2$, $\vec{\zeta} = (0.2, 0)$, $\vec{\lambda} = (1, 10)$.
The high fidelity means were chosen to be a uniform grid in $(0.1, 0.9)$. The Gaussian distributions had standard deviation 1.
4. Bernoulli: $M = 1000$, $M = 5$, $\vec{\zeta} = (0.5, 0.2, 0.1, 0.05, 0)$, $\vec{\lambda} = (1, 3, 10, 30, 100)$.
The high fidelity means were chosen to be a uniform grid in $(0.1, 0.9)$. The Gaussian distributions had standard deviation 1.

In all cases above, the lower fidelity means were sampled uniformly within a $\pm \zeta^{(m)}$ band around $\mu_k^{(M)}$. In addition, for the Gaussian distributions we modified the lower fidelity means of the optimal arm $\mu_{k_\star}^{(m)}$, $m < M$ to be lower than the corresponding mean of a suboptimal arm. For the Bernoulli

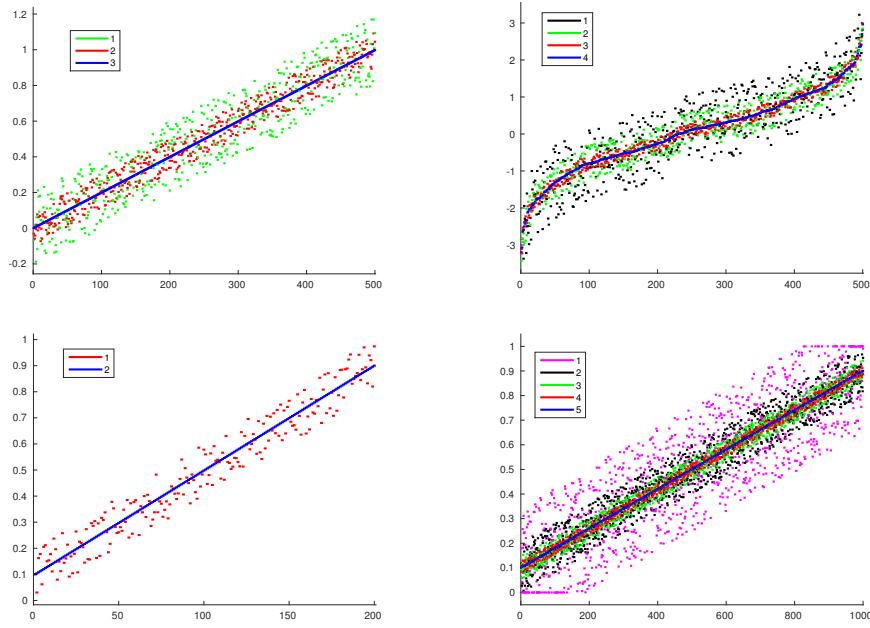


Figure 3: An illustration of the means of the arms used the simulation problems. The top row are the Gaussian rewards with (K, M) equal to $(500, 3)$, $(500, 4)$ while the second row are the Bernoulli rewards with $(200, 2)$, $(1000, 5)$ respectively.

rewards, if $\mu_k^{(m)}$ fell outside of $(0, 1)$ its value was truncated. Figure 3 illustrates the mean values of these arms.

For both MF-UCB and UCB we used $\rho = 2$ [5].

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