

## A Technical Lemmas

**Lemma 1.** For all episodes  $t$  and  $k \leq K$ :

$$\mathbb{E}_{\phi_t} [\mathbb{1}_{i,k,t}(\phi_t)(F_{k \rightarrow}^g(\phi_t) - F_{k \rightarrow}^t(\phi_t))] \leq \mathbb{E}_{\phi_t} [\mathbb{1}_{i,k,t}(\phi_t)] G_k.$$

*Proof.* The first term in the expectation can be written as:

$$\begin{aligned} \mathbb{E}_{\phi_t} [\mathbb{1}_{i,k,t}(\phi_t) F_{k \rightarrow}^g(\phi_t)] &\stackrel{(a)}{=} \sum_{\mathbf{y} \in \mathcal{Y}_k} P(\phi_t \sim \mathbf{y}) \mathbb{E}_{\phi_t} [\mathbb{1}_{i,k,t}(\phi_t) F_{k \rightarrow}^g(\phi_t) \mid \phi_t \sim \mathbf{y}] \\ &\stackrel{(b)}{=} \sum_{\mathbf{y} \in \mathcal{Y}_k} P(\phi_t \sim \mathbf{y}) \mathbb{1}_{i,k,t}(\mathbf{y}) \mathbb{E}_{\phi_t} [F_{k \rightarrow}^g(\phi_t) \mid \phi_t \sim \mathbf{y}]. \end{aligned} \quad (21)$$

The equality (a) is due to conditioning  $\phi_t$  on the first  $k-1$  observations  $\mathbf{y}$  under the policy  $\pi^g$ . The equality (b) follows from the fact that the indicator  $\mathbb{1}_{i,k,t}(\phi_t)$  (Equation 15) does not depend on the observations from step  $k$  forward. The last two terms in Equation 21 may depend on each other in a non-trivial manner. To break this dependency, we first bound the expectation over  $\mathbf{y}$  using Hölder's inequality and then the latter term by Lemma 2:

$$\begin{aligned} \mathbb{E}_{\phi_t} [\mathbb{1}_{i,k,t}(\phi_t) F_{k \rightarrow}^g(\phi_t)] &\leq \mathbb{E}_{\phi_t} [\mathbb{1}_{i,k,t}(\phi_t)] \max_{\mathbf{y} \in \mathcal{Y}_k} \mathbb{E}_{\phi_t} [F_{k \rightarrow}^g(\phi_t) \mid \phi_t \sim \mathbf{y}] \\ &\leq \mathbb{E}_{\phi_t} [\mathbb{1}_{i,k,t}(\phi_t)] G_k, \end{aligned} \quad (22)$$

where  $G_k = (K - k + 1) \max_{\mathbf{y} \in \mathcal{Y}_k} \max_i g_i(\mathbf{y})$  is an upper bound on the expected gain of  $\pi^g$  from step  $k$  forward. The term  $G_k$  is independent of  $\phi_t$  and  $\pi_t$ , the state and policy in episode  $t$ .

Similarly to Equation 21, the second term in the expectation can be written as:

$$\mathbb{E}_{\phi_t} [\mathbb{1}_{i,k,t}(\phi_t) F_{k \rightarrow}^t(\phi_t)] = \sum_{\mathbf{y} \in \mathcal{Y}_k^t} P(\phi_t \sim \mathbf{y}) \mathbb{1}_{i,k,t}(\mathbf{y}) \mathbb{E}_{\phi_t} [F_{k \rightarrow}^t(\phi_t) \mid \phi_t \sim \mathbf{y}]. \quad (23)$$

In Lemma 3, we show that  $\mathbb{E}_{\phi_t} [F_{k \rightarrow}^t(\phi_t) \mid \phi_t \sim \mathbf{y}] \geq 0$  for all  $\mathbf{y}$  and  $k$ . It follows that:

$$-\mathbb{E}_{\phi_t} [\mathbb{1}_{i,k,t}(\phi_t) F_{k \rightarrow}^t(\phi_t)] \leq 0. \quad (24)$$

Our main claim is obtained by combining the upper bounds in Equations 22 and 24. ■

**Lemma 2.** For all  $k \leq K$ :

$$\max_{\mathbf{y} \in \mathcal{Y}_k} \mathbb{E}_{\phi} [F_{k \rightarrow}^g(\phi) \mid \phi \sim \mathbf{y}] \leq (K - k + 1) \max_{\mathbf{y} \in \mathcal{Y}_k} \max_i g_i(\mathbf{y}).$$

*Proof.* First, we note that for all contexts  $\mathbf{y} \in \mathcal{Y}_k$ :

$$\begin{aligned} \mathbb{E}_{\phi} [F_{k \rightarrow}^g(\phi) \mid \phi \sim \mathbf{y}] &= \sum_{j=k}^K \mathbb{E}_{\phi} [f(\pi_j^g(\phi), \phi) - f(\pi_{j-1}^g(\phi), \phi) \mid \phi \sim \mathbf{y}] \\ &= \sum_{j=k}^K \sum_{\mathbf{y}' \in \mathcal{Y}_j} P(\phi \sim \mathbf{y}' \mid \phi \sim \mathbf{y}) \mathbb{E}_{\phi} [f(\pi_j^g(\phi), \phi) - f(\pi_{j-1}^g(\phi), \phi) \mid \phi \sim \mathbf{y}, \phi \sim \mathbf{y}'] \\ &= \sum_{j=k}^K \sum_{\mathbf{y}' \in \mathcal{Y}_j} P(\phi \sim \mathbf{y}' \mid \phi \sim \mathbf{y}) \mathbb{E}_{\phi} [f(\pi_j^g(\phi), \phi) - f(\pi_{j-1}^g(\phi), \phi) \mid \phi \sim \mathbf{y}']. \end{aligned} \quad (25)$$

The last equality follows from the fact that  $P(\phi \sim \mathbf{y}' \mid \phi \sim \mathbf{y}) > 0$  implies  $\mathbf{y}' \succeq \mathbf{y}$ , and this further implies that  $(\phi \sim \mathbf{y}) \wedge (\phi \sim \mathbf{y}') \equiv \phi \sim \mathbf{y}'$ . Second, because the policy  $\pi^g$  in context  $\mathbf{y}$  chooses an item with the largest expected gain and  $f$  is adaptive submodular, we know that:

$$\mathbb{E}_{\phi} [f(\pi_j^g(\phi), \phi) - f(\pi_{j-1}^g(\phi), \phi) \mid \phi \sim \mathbf{y}'] \leq \max_i g_i(\mathbf{y}) \quad (26)$$

for all  $j$  and  $\mathbf{y}' \succeq \mathbf{y}$ . This upper bound can be substituted into Equation 25 and yields:

$$\mathbb{E}_{\phi} [F_{k \rightarrow}^g(\phi) \mid \phi \sim \mathbf{y}] \leq (K - k + 1) \max_i g_i(\mathbf{y}). \quad (27)$$

Our main claim is obtained by maximizing both sides of the above inequality over  $\mathbf{y}$ . ■

**Lemma 3.** For all  $k \leq K$  and  $\mathbf{y} \in \mathcal{Y}_k^t$ :

$$\mathbb{E}_\phi [F_{k \rightarrow}^t(\phi) \mid \phi \sim \mathbf{y}] \geq 0.$$

*Proof.* Similarly to Lemma 2, we note that for all  $\mathbf{y} \in \mathcal{Y}_k^t$ :

$$\begin{aligned} & \mathbb{E}_\phi [F_{k \rightarrow}^t(\phi) \mid \phi \sim \mathbf{y}] \\ &= \sum_{j=k}^K \mathbb{E}_\phi [f(\pi_j^t(\phi), \phi) - f(\pi_{j-1}^t(\phi), \phi) \mid \phi \sim \mathbf{y}] \\ &= \sum_{j=k}^K \sum_{\mathbf{y}' \in \mathcal{Y}_j^t} P(\phi \sim \mathbf{y}' \mid \phi \sim \mathbf{y}) \mathbb{E}_\phi [f(\pi_j^t(\phi), \phi) - f(\pi_{j-1}^t(\phi), \phi) \mid \phi \sim \mathbf{y}, \phi \sim \mathbf{y}'] \\ &= \sum_{j=k}^K \sum_{\mathbf{y}' \in \mathcal{Y}_j^t} P(\phi \sim \mathbf{y}' \mid \phi \sim \mathbf{y}) \mathbb{E}_\phi [f(\pi_j^t(\phi), \phi) - f(\pi_{j-1}^t(\phi), \phi) \mid \phi \sim \mathbf{y}']. \end{aligned} \quad (28)$$

The last equality follows from the fact that  $P(\phi \sim \mathbf{y}' \mid \phi \sim \mathbf{y}) > 0$  implies  $\mathbf{y}' \succeq \mathbf{y}$ , and this further implies that  $(\phi \sim \mathbf{y}) \wedge (\phi \sim \mathbf{y}') \equiv \phi \sim \mathbf{y}'$ . Because  $f$  is adaptive monotonic, we know that:

$$\mathbb{E}_\phi [f(\pi_j^t(\phi), \phi) - f(\pi_{j-1}^t(\phi), \phi) \mid \phi \sim \mathbf{y}'] \geq 0 \quad (29)$$

for all  $j$  and  $\mathbf{y}'$ . Our main claim follows from substituting the above bound into Equation 28. ■

**Lemma 4.** For all arms  $i$  and  $k \leq K$ :

$$\mathbb{E}_{\phi_1, \dots, \phi_n} \left[ \sum_{t=1}^n \mathbb{1}_{i,k,t}(\phi_t) \mathbb{1}\{T_i(t-1) > \ell_{i,k}\} \right] \leq \frac{2}{3} \pi^2 (L+1), \quad (30)$$

$$\text{where } \ell_{i,k} = \left\lceil 8 \max_{\mathbf{y} \in \mathcal{Y}_{k,i}} \frac{\bar{g}_i^2(\mathbf{y})}{\Delta_i^2(\mathbf{y})} \log n \right\rceil.$$

*Proof.* Our proof has the same structure as the proof of Theorem 1 by Auer *et al.* [1]. Let  $\ell_{i,k}$  be a positive integer. Then for all arms  $i$  and steps  $k$ :

$$\begin{aligned} & \sum_{t=1}^n \mathbb{1}_{i,k,t}(\phi_t) \mathbb{1}\{T_i(t-1) > \ell_{i,k}\} \\ &= \sum_{t=\ell_{i,k}+1}^n \mathbb{1}_{i,k,t}(\phi_t) \mathbb{1}\{T_i(t-1) > \ell_{i,k}\} \\ &\leq \sum_{t=\ell_{i,k}+1}^n \mathbb{1}\{\exists \mathbf{y} \in \mathcal{Y}_{k,i} : (\hat{p}_{i,T_i(t-1)} + c_{t-1,T_i(t-1)}) \bar{g}_i(\mathbf{y}) \geq \\ &\quad (\hat{p}_{i^*(\mathbf{y}),T_{i^*(\mathbf{y})}(t-1)} + c_{t-1,T_{i^*(\mathbf{y})}(t-1)}) \bar{g}_{i^*(\mathbf{y})}(\mathbf{y}), T_i(t-1) > \ell_{i,k}\} \\ &\leq \sum_{t=\ell_{i,k}+1}^n \sum_{s=1}^t \sum_{s_i=\ell_{i,k}+1}^t \mathbb{1}\{\exists \mathbf{y} \in \mathcal{Y}_{k,i} : (\hat{p}_{i,s_i} + c_{t-1,s_i}) \bar{g}_i(\mathbf{y}) \geq (\hat{p}_{i^*(\mathbf{y}),s} + c_{t-1,s}) \bar{g}_{i^*(\mathbf{y})}(\mathbf{y})\} \\ &= \sum_{t=\ell_{i,k}}^{n-1} \sum_{s=1}^{t+1} \sum_{s_i=\ell_{i,k}+1}^{t+1} \mathbb{1}\{\exists \mathbf{y} \in \mathcal{Y}_{k,i} : (\hat{p}_{i,s_i} + c_{t,s_i}) \bar{g}_i(\mathbf{y}) \geq (\hat{p}_{i^*(\mathbf{y}),s} + c_{t,s}) \bar{g}_{i^*(\mathbf{y})}(\mathbf{y})\}. \end{aligned} \quad (31)$$

The existence of  $\mathbf{y} \in \mathcal{Y}_{k,i}$  such that  $(\hat{p}_{i,s_i} + c_{t,s_i}) \bar{g}_i(\mathbf{y}) \geq (\hat{p}_{i^*(\mathbf{y}),s} + c_{t,s}) \bar{g}_{i^*(\mathbf{y})}(\mathbf{y})$  implies that at least one of the following claims must be true:

$$\exists \mathbf{y} \in \mathcal{Y}_{k,i} : \hat{p}_{i^*(\mathbf{y}),s} \leq p_{i^*(\mathbf{y})} - c_{t,s} \quad (32)$$

$$\hat{p}_{i,s_i} \geq p_i + c_{t,s_i} \quad (33)$$

$$\exists \mathbf{y} \in \mathcal{Y}_{k,i} : p_{i^*(\mathbf{y})} \bar{g}_{i^*(\mathbf{y})}(\mathbf{y}) < p_i \bar{g}_i(\mathbf{y}) + 2c_{t,s_i} \bar{g}_i(\mathbf{y}). \quad (34)$$

We bound the probability of the first two events (Equations 32 and 33) using Hoeffding's inequality and the union bound:

$$P(\exists \mathbf{y} \in \mathcal{Y}_{k,i} : \hat{p}_{i^*}(\mathbf{y})_{,s} \leq p_{i^*}(\mathbf{y})_{,s} - c_{t,s}) \leq L \exp[-4 \log t] = Lt^{-4} \quad (35)$$

$$P(\hat{p}_{i,s_i} \geq p_i + c_{t,s_i}) \leq \exp[-4 \log t] = t^{-4}. \quad (36)$$

When  $\ell_{i,k} = \left\lceil 8 \max_{\mathbf{y} \in \mathcal{Y}_{k,i}} \frac{\bar{g}_i^2(\mathbf{y})}{\Delta_i^2(\mathbf{y})} \log n \right\rceil$ , the third event (Equation 34) cannot happen. In particular, for all  $\mathbf{y} \in \mathcal{Y}_{k,i}$  and  $s_i \geq 8 \max_{\mathbf{y} \in \mathcal{Y}_{k,i}} \frac{\bar{g}_i^2(\mathbf{y})}{\Delta_i^2(\mathbf{y})} \log n$ , we can show that:

$$\begin{aligned} p_{i^*}(\mathbf{y}) \bar{g}_{i^*}(\mathbf{y})(\mathbf{y}) - p_i \bar{g}_i(\mathbf{y}) - 2c_{t,s_i} \bar{g}_i(\mathbf{y}) &= \bar{g}_i(\mathbf{y}) \left[ \frac{\Delta_i(\mathbf{y})}{\bar{g}_i(\mathbf{y})} - 2\sqrt{\frac{2 \log t}{s_i}} \right] \\ &\geq \bar{g}_i(\mathbf{y}) \left[ \frac{\Delta_i(\mathbf{y})}{\bar{g}_i(\mathbf{y})} - \min_{\mathbf{y} \in \mathcal{Y}_{k,i}} \frac{\Delta_i(\mathbf{y})}{\bar{g}_i(\mathbf{y})} \right] \\ &\geq 0. \end{aligned} \quad (37)$$

Therefore, we may conclude that:

$$\begin{aligned} &\mathbb{E}_{\phi_1, \dots, \phi_n} \left[ \sum_{t=1}^n \mathbb{1}_{i,k,t}(\phi_t) \mathbb{1}\{T_i(t-1) > \ell_{i,k}\} \right] \\ &\leq \sum_{t=1}^{\infty} \sum_{s=1}^{t+1} \sum_{s_i=1}^{t+1} [P(\exists \mathbf{y} \in \mathcal{Y}_{k,i} : \hat{p}_{i^*}(\mathbf{y})_{,s} \leq p_{i^*}(\mathbf{y})_{,s} - c_{t,s}) + P(\hat{p}_{i,s_i} \geq p_i + c_{t,s_i})] \\ &\leq (L+1) \sum_{t=1}^{\infty} (t+1)^2 t^{-4} \\ &\leq (L+1) \sum_{t=1}^{\infty} 4t^{-2} \\ &= \frac{2}{3} \pi^2 (L+1). \end{aligned} \quad (38)$$

■

## B Categorical State Variables

In this section, we show how to generalize our work to categorical state variables.

We assume that each item  $i$  has  $M$  possible states,  $\phi[i] \in \{1, \dots, M\}$ . The state of item  $i$  is drawn i.i.d. from a categorical distribution, which is described by  $M$  probabilities  $p_{i,1}, \dots, p_{i,M}$  such that  $\sum_{m=1}^M p_{i,m} = 1$ . In this setting, the joint probability distribution of states factors as:

$$P(\Phi = \phi) = \prod_{i=1}^L \prod_{m=1}^M p_{i,m}^{\mathbb{1}\{\phi[i]=m\}}. \quad (39)$$

Based on the above assumption, we rewrite the expected gain (Equation 5) as:

$$g_i(\mathbf{y}) = \sum_{m=1}^M p_{i,m} \bar{g}_{i,m}(\mathbf{y}), \quad (40)$$

where:

$$\bar{g}_{i,m}(\mathbf{y}) = \mathbb{E}_{\phi} [f(\text{dom}(\mathbf{y}) \cup \{i\}, \phi) - f(\text{dom}(\mathbf{y}), \phi) \mid \phi \sim \mathbf{y}, \phi[i] = m] \quad (41)$$

is the expected gain when item  $i$  is in state  $m$ . Similarly to Section 3.1, we assume that the function  $\bar{g}_{i,m}(\mathbf{y})$  is known and can be computed without knowing  $P(\Phi)$ .

Algorithm OASM changes in the computation of the index. The index is computed as:

$$\sum_{m=1}^M (\hat{p}_{i,m,T_i(t-1)} + c_{t-1,T_i(t-1)}) \bar{g}_{i,m}(\mathbf{y}), \quad (42)$$

where  $\hat{p}_{i,m,T_i(t-1)}$  is the maximum-likelihood estimate of  $p_{i,m}$  from the first  $t - 1$  episodes, which is computed from  $T_i(t - 1)$  observations of item  $i$ .

Our analysis changes in Lemma 4. First, the events in Equations 32 and 33 have to be bounded for all  $m \in \{1, \dots, M\}$ . Second, the event in Equation 34 does not happen when:

$$\ell_{i,k} = \left\lceil 8 \max_{\mathbf{y} \in \mathcal{Y}_{k,i}} \frac{\left( \sum_{m=1}^M \bar{g}_{i,m}(\mathbf{y}) \right)^2}{\Delta_i^2(\mathbf{y})} \log n \right\rceil. \quad (43)$$

As a result, our final regret bound is:

$$R(n) \leq \underbrace{\sum_{i=1}^L \ell_i \sum_{k=1}^K G_k \alpha_{i,k}}_{O(\log n)} + \underbrace{\frac{2}{3} \pi^2 M L (L + 1) \sum_{k=1}^K G_k}_{O(1)}. \quad (44)$$