
Appendix to Super-Bit Locality-Sensitive Hashing

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1 Proof of Lemma 6

Proof. For the first part, by Lemma 5, we only need to prove $Pr[X_i = 1|X_1 = 1] = Pr[X_2 = 1|X_1 = 1]$.

$$Pr[X_i = 1|X_1 = 1] = \int Pr[X_i = 1|X_1 = 1, w_1]p(w_1|X_1 = 1)dw_1.$$

Without loss of generality, let $\|w_1\| = 1$. And expand w_1 to $[w_1, B_1]$ whose columns constitute the orthonormal bases of \mathbb{R}^d . Denote $\tilde{a} = B_1^T a$, $\tilde{b} = B_1^T b$, and $\tilde{w}_i = B_1^T w_i$, for $2 \leq i \leq d$. Since $w_i = B_1 B_1^T w_i = B_1 \tilde{w}_i$, we have $w_i^T a = \tilde{w}_i^T \tilde{a}$ and $w_i^T b = \tilde{w}_i^T \tilde{b}$, $2 \leq i \leq d$. Then

$$Pr[X_i = 1|X_1 = 1, w_1] = Pr[\text{sgn}(w_i^T a) \neq \text{sgn}(w_i^T b)|X_1 = 1, w_1] = Pr[\text{sgn}(\tilde{w}_i^T \tilde{a}) \neq \text{sgn}(\tilde{w}_i^T \tilde{b})|X_1 = 1, w_1] = Pr[\text{sgn}(\tilde{w}_i^T \tilde{a}) \neq \text{sgn}(\tilde{w}_i^T \tilde{b})|w_1, a^T w_1 b^T w_1 < 0].$$

Therefore we only need to consider the subproblem: In a given $(d-1)$ -dimensional subspace S_1^\perp , given data samples \tilde{a}, \tilde{b} , compute $Pr[h_{\tilde{w}_i}(\tilde{a}) \neq h_{\tilde{w}_i}(\tilde{b})|w_1, a^T w_1 b^T w_1 < 0]$. Let $\tilde{v}_2 = B_1^T v_2, \dots, \tilde{v}_N = B_1^T v_N$. It can be shown that given $w_1, \{\tilde{v}_2, \dots, \tilde{v}_N\}$ are independent isotropic normal vectors in subspace S_1^\perp , and $\tilde{w}_2, \dots, \tilde{w}_N$ are the result of Gram-Schmidt process applying to $\{\tilde{v}_2, \dots, \tilde{v}_N\}$. Apply Theorem 1 to this subproblem, we have $Pr[h_{\tilde{w}_i}(\tilde{a}) \neq h_{\tilde{w}_i}(\tilde{b})|w_1, a^T w_1 b^T w_1 < 0] = \theta_{\tilde{a}, \tilde{b}}/\pi$. Since $\theta_{\tilde{a}, \tilde{b}}$ is a function of a, b, w_1 , $Pr[X_i = 1|X_1 = 1, w_1] = \theta_{\tilde{a}, \tilde{b}}/\pi = Pr[X_2 = 1|X_1 = 1, w_1]$, and $Pr[X_i = 1|X_1 = 1] = Pr[X_2 = 1|X_1 = 1]$.

For the second part it suffices to prove that if $\theta_{a,b} \in (0, \frac{\pi}{2}]$, $\theta_{\tilde{a}, \tilde{b}} < \theta_{a,b}$, which is equal to prove that $\cos \theta_{a,b} = \frac{a^T b}{\|a\| \|b\|} < \frac{\tilde{a}^T \tilde{b}}{\|\tilde{a}\| \|\tilde{b}\|} = \frac{a^T b - a^T w_1 w_1^T b}{\|a - w_1 w_1^T a\| \|b - w_1 w_1^T b\|} = \cos \theta_{\tilde{a}, \tilde{b}}$. Since $a^T b \geq 0$, $(a^T w_1)(b^T w_1) < 0$, we have $\cos \theta_{\tilde{a}, \tilde{b}} > 0$, thus it is equal to prove that $\cos^2 \theta_{\tilde{a}, \tilde{b}} - \cos^2 \theta_{a,b} > 0$, which is again equal to prove that $(a^T b - a^T w_1 w_1^T b)^2 \|a\|^2 \|b\|^2 - (a^T b)^2 \|a - w_1 w_1^T a\|^2 \|b - w_1 w_1^T b\|^2 > 0$. After some algebraic calculations we get that the left side equals to $(\|a\|^2 \|b\|^2 - (a^T b)^2)(a^T w_1)^2 (b^T w_1)^2 + a^T b \|a\|^2 (a^T b (b^T w_1)^2 - \|b\|^2 b^T w_1 a^T w_1) + a^T b \|b\|^2 (a^T b (a^T w_1)^2 - \|a\|^2 a^T w_1 b^T w_1)$.

Since $a^T b \geq 0$, $a^T w_1 b^T w_1 < 0$, $\|a\|^2 \|b\|^2 > (a^T b)^2$, the expression above is positive. Thus we have proved that $\theta_{\tilde{a}, \tilde{b}} < \theta_{a,b}$.

Therefore if $\theta_{a,b} \in (0, \frac{\pi}{2}]$, $Pr[X_2 = 1|X_1 = 1] = \int Pr[X_2 = 1|X_1 = 1, w_1]p(w_1|X_1 = 1)dw_1 = \int \frac{\theta_{\tilde{a}, \tilde{b}}}{\pi} p(w_1|X_1 = 1)dw_1 < \int \frac{\theta_{a,b}}{\pi} p(w_1|X_1 = 1)dw_1 = \frac{\theta_{a,b}}{\pi}$ \square