Conditional Random Sampling: A Sketch-based Sampling Technique for Sparse Data

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Abstract

We[1] develop *Conditional Random Sampling (CRS)*, a technique particularly suitable for sparse data. In large-scale applications, the data are often highly sparse. CRS combines sketching and sampling in that it converts sketches of the data into *conditional random samples* online in the estimation stage, with the sample size determined retrospectively. This paper focuses on approximating pairwise l_2 and l_1 distances and comparing CRS with random projections. For boolean (0/1) data, CRS is provably better than random projections. We show using real-world data that CRS often outperforms random projections. This technique can be applied in learning, data mining, information retrieval, and database query optimizations.

1 Introduction

Conditional Random Sampling (CRS) is a sketch-based sampling technique that effectively exploits data sparsity. In modern applications in learning, data mining, and information retrieval, the datasets are often very large and also highly sparse. For example, the *term-document* matrix is often more than 99% sparse [7]. Sampling large-scale sparse data is challenging. The *conventional random sampling* (i.e., randomly picking a small fraction) often performs poorly when most of the samples are zeros. Also, in heavy-tailed data, the estimation errors of random sampling could be very large.

As alternatives to random sampling, various *sketching* algorithms have become popular, e.g., random projections [17] and min-wise sketches [6]. Sketching algorithms are designed for approximating *specific* summary statistics. For a specific task, a sketching algorithm often outperforms random sampling. On the other hand, random sampling is much more flexible. For example, we can use the same set of random samples to estimate any l_p pairwise distances and multi-way associations. *Conditional Random Sampling (CRS)* combines the advantages of both sketching and random sampling.

Many important applications concern only the pairwise distances, e.g., distance-based clustering and classification, multi-dimensional scaling, kernels. For a large training set (e.g., at Web scale), computing pairwise distances exactly is often too time-consuming or even infeasible.

Let $\mathbf{A}$ be a data matrix of n rows and D columns. For example, $\mathbf{A}$ can be the *term-document* matrix with n as the total number of word types and D as the total number of documents. In modern search engines, $n \approx 10^6 \sim 10^7$ and $D \approx 10^{10} \sim 10^{11}$. In general, n is the number of data points and D is the number of features. Computing all pairwise associations $\mathbf{A}\mathbf{A}^{\mathrm{T}}$, also called the *Gram matrix* in machine learning, costs $O(n^2D)$, which could be daunting for large n and D. Various sampling methods have been proposed for approximating Gram matrix and kernels [2, 8]. For example, using (normal) random projections [17], we approximate $\mathbf{A}\mathbf{A}^{\mathrm{T}}$ by $(\mathbf{A}\mathbf{R})(\mathbf{A}\mathbf{R})^{\mathrm{T}}$, where the entries of $\mathbf{R} \in \mathbb{R}^{D\times k}$ are i.i.d. $N(0,1)$. This reduces the cost down to $O(nDk+n^2k)$, where $k \ll \min(n, D)$.

[1]The full version [13]: www.stanford.edu/~pingli98/publications/CRS_tr.pdf

Sampling techniques can be critical in databases and information retrieval. For example, the database query optimizer seeks highly efficient techniques to estimate the intermediate join sizes in order to choose an "optimum" execution path for multi-way joins.

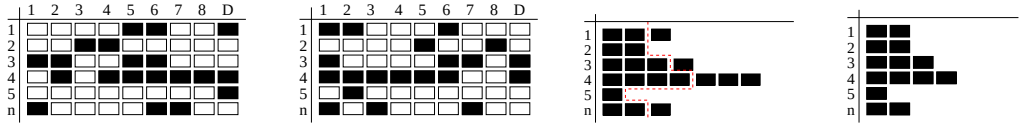
Conditional Random Sampling (CRS) can be applied to estimating pairwise distances (in any norm) as well as multi-way associations. CRS can also be used for estimating joint histograms (two-way and multi-way). While this paper focuses on estimating pairwise l_2 and l_1 distances and inner products, we refer readers to the technical report [13] for estimating joint histograms. Our early work, [11, 12] concerned estimating two-way and multi-way associations in boolean (0/1) data.

We will compare CRS with *normal random projections* for approximating l_2 distances and inner products, and with *Cauchy random projections* for approximating l_1 distances. In boolean data, CRS bears some similarity to *Broder's sketches* [6] with some important distinctions. [12] showed that in boolean data, CRS improves Broder's sketches by roughly halving the estimation variances.

2 The Procedures of CRS

Conditional Random Sampling is a two-stage procedure. In the *sketching* stage, we scan the data matrix once and store a fraction of the non-zero elements in each data point, as "sketches." In the *estimation* stage, we generate *conditional random samples* online pairwise (for two-way) or group-wise (for multi-way); hence we name our algorithm *Conditional Random Sampling (CRS)*.

2.1 The Sampling/Sketching Procedure



(a) Original (b) Permuted (c) Postings (d) Sketches

Figure 1: A global view of the *sketching* stage.

Figure 1 provides a global view of the *sketching* stage. The columns of a sparse data matrix (a) are first randomly permuted (b). Then only the non-zero entries are considered, called *postings* (c). *Sketches* are simply the front of postings (d). Note that in the actual implementation, we only need to maintain a permutation mapping on the column IDs.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
u_1	0	1	0	2	0	1	0	0	1	2	1	0	1	0	2
u_2	1	3	0	0	1	2	0	1	0	0	3	0	0	2	1

(a) Data matrix and random samples

P_1: 2 (1) 4 (2) 6 (1) 9 (1) 10 (2) 11 (1) 13 (1) 15 (2)
P_2: 1 (1) 2 (3) 5 (1) 6 (2) 8 (1) 11 (3) 14 (2) 15 (1)

(b) Postings

K_1: 2 (1) 4 (2) 6 (1) 9 (1) 10 (2)
K_2: 1 (1) 2 (3) 5 (1) 6 (2) 8 (1) 11 (3)

(c) Sketches

Figure 2: (a): A data matrix with two rows and $D = 15$. If the column IDs are random, the first $D_s = 10$ columns constitute a random sample. u_i denotes the ith row. (b): Postings consist of tuples "ID (Value)." (c): Sketches are the first k_i entries of postings sorted ascending by IDs. In this example, $k_1 = 5$, $k_2 = 6$, $D_s = \min(10, 11) = 10$. Excluding 11(3) in K_2, we obtain the same samples as if we directly sampled the first $D_s = 10$ columns in the data matrix.

Apparently sketches are not uniformly random samples, which may make the estimation task difficult. We show, in Figure 2, that sketches are almost random samples pairwise (or group-wise). Figure 2(a) constructs *conventional random samples* from a data matrix; and we show one can generate (retrospectively) the same random samples from sketches in Figure 2(b)(c).

In Figure 2(a), when the column are randomly permuted, we can construct random samples by simply taking the first D_s columns from the data matrix of D columns ($D_s \ll D$ in real applications).

For sparse data, we only store the non-zero elements in the form of tuples "ID (Value)," a structure called *postings*. We denote the postings by P_i for each row u_i. Figure 2(b) shows the postings for the same data matrix in Figure 2(a). The tuples are sorted ascending by their IDs. A *sketch*, K_i, of postings P_i, is the first k_i entries (i.e., the smallest k_i IDs) of P_i, as shown in Figure 2(c).

The central observation is that if we exclude all elements of sketches whose IDs are larger than

$$D_s = \min\left(\max(\text{ID}(\text{K}_1)), \max(\text{ID}(\text{K}_2))\right), \tag{1}$$

we obtain exactly the same samples as if we directly sampled the first D_s columns from the data matrix in Figure 2(a). This way, we convert sketches into random samples by *conditioning* on D_s, which differs pairwise and we do not know beforehand.

2.2 The Estimation Procedure

The estimation task for CRS can be extremely simple. After we construct the conditional random samples from sketches K_1 and K_2 with the effective sample size D_s, we can compute any distances (l_2, l_1, or inner products) from the samples and multiply them by $\frac{D}{D_s}$ to estimate the original space. (Later, we will show how to improve the estimates by taking advantage of the marginal information.)

We use $\tilde{u}_{1,j}$ and $\tilde{u}_{2,j}$ ($j = 1$ to D_s) to denote the conditional random samples (of size D_s) obtained by CRS. For example, in Figure 2, we have $D_s = 10$, and the non-zero $\tilde{u}_{1,j}$ and $\tilde{u}_{2,j}$ are

$$\tilde{u}_{1,2} = 3,\ \tilde{u}_{1,4} = 2,\ \tilde{u}_{1,6} = 1,\ \tilde{u}_{1,9} = 1,\ \tilde{u}_{1,10} = 2$$
$$\tilde{u}_{2,1} = 1,\ \tilde{u}_{2,2} = 3,\ \tilde{u}_{2,5} = 1,\ \tilde{u}_{2,6} = 2,\ \tilde{u}_{2,8} = 1.$$

Denote the inner product, squared l_2 distance, and l_1 distance, by a, $d^{(2)}$, and $d^{(1)}$, respectively,

$$a = \sum_{i=1}^{D} u_{1,i}u_{2,i}, \qquad d^{(2)} = \sum_{i=1}^{D} |u_{1,i} - u_{2,i}|^2, \qquad d^{(1)} = \sum_{i=1}^{D} |u_{1,i} - u_{2,i}| \tag{2}$$

Once we have the random samples, we can then use the following simple linear estimators:

$$\hat{a}_{MF} = \frac{D}{D_s}\sum_{j=1}^{D_s} \tilde{u}_{1,j}\tilde{u}_{2,j}, \quad \hat{d}^{(2)}_{MF} = \frac{D}{D_s}\sum_{j=1}^{D_s} \left(\tilde{u}_{1,j} - \tilde{u}_{2,j}\right)^2, \quad \hat{d}^{(1)}_{MF} = \frac{D}{D_s}\sum_{j=1}^{D_s} |\tilde{u}_{1,j} - \tilde{u}_{2,j}|. \tag{3}$$

2.3 The Computational Cost

Th sketching stage requires generating a random permutation mapping of length D, and linear scan all the non-zeros. Therefore, generating sketches for $\mathbf{A} \in \mathbb{R}^{n\times D}$ costs $O(\sum_{i=1}^{n} f_i)$, where f_i is the number of non-zeros in the ith row, i.e., $f_i = |\text{P}_i|$. In the estimation stage, we need to linear scan the sketches. While the conditional sample size D_s might be large, the cost for estimating the distance between one pair of data points would be only $O(k_1 + k_2)$ instead of $O(D_s)$.

3 The Theoretical Variance Analysis of CRS

We give some theoretical analysis on the variances of CRS. For simplicity, we ignore the "finite population correction factor", $\frac{D-D_s}{D-1}$, due to "sample-without-replacement."

We first consider $\hat{a}_{MF} = \frac{D}{D_s}\sum_{j=1}^{D_s} \tilde{u}_{1,j}\tilde{u}_{2,j}$. By assuming "sample-with-replacement," the samples, $(\tilde{u}_{1,j}\tilde{u}_{2,j})$, $j = 1$ to D_s, are i.i.d, conditional on D_s. Thus,

$$\text{Var}(\hat{a}_{MF}|D_s) = \left(\frac{D}{D_s}\right)^2 D_s \text{Var}\left(\tilde{u}_{1,1}\tilde{u}_{2,1}\right) = \frac{D}{D_s} D\left(\text{E}\left(\tilde{u}_{1,1}\tilde{u}_{2,1}\right)^2 - \text{E}^2\left(\tilde{u}_{1,1}\tilde{u}_{2,1}\right)\right), \tag{4}$$

$$\text{E}\left(\tilde{u}_{1,1}\tilde{u}_{2,1}\right) = \frac{1}{D}\sum_{i=1}^{D}\left(u_{1,i}u_{2,i}\right) = \frac{a}{D}, \qquad \text{E}\left(\tilde{u}_{1,1}\tilde{u}_{2,1}\right)^2 = \frac{1}{D}\sum_{i=1}^{D}\left(u_{1,i}u_{2,i}\right)^2, \tag{5}$$

$$\text{Var}(\hat{a}_{MF}|D_s) = \frac{D}{D_s} D\left(\frac{1}{D}\sum_{i=1}^{D}\left(u_{1,i}u_{2,i}\right)^2 - \left(\frac{a}{D}\right)^2\right) = \frac{D}{D_s}\left(\sum_{i=1}^{D} u_{1,i}^2 u_{2,i}^2 - \frac{a^2}{D}\right). \tag{6}$$

The unconditional variance would be simply

$$\text{Var}(\hat{a}_{MF}) = \text{E}\left(\text{Var}(\hat{a}_{MF}|D_s)\right) = \text{E}\left(\frac{D}{D_s}\right)\left(\sum_{i=1}^{D} u_{1,i}^2 u_{2,i}^2 - \frac{a^2}{D}\right),$$

as $\text{Var}(\hat{X}) = \text{E}\left(\text{Var}(\hat{X}|D_s)\right) + \text{Var}\left(\text{E}(\hat{X}|D_s)\right) = \text{E}\left(\text{Var}(\hat{X}|D_s)\right)$, when $\hat{X}$ is conditionally unbiased.

No closed-form expression is known for $\text{E}\left(\frac{D}{D_s}\right)$; but we know $\text{E}\left(\frac{D}{D_s}\right) \geq \max\left(\frac{f_1}{k_1}, \frac{f_2}{k_2}\right)$ (similar to Jensen's inequality). Asymptotically (as k_1 and k_2 increase), the inequality becomes an equality

$$\text{E}\left(\frac{D}{D_s}\right) \approx \max\left(\frac{f_1+1}{k_1}, \frac{f_2+1}{k_2}\right) \approx \max\left(\frac{f_1}{k_1}, \frac{f_2}{k_2}\right), \tag{7}$$

where f_1 and f_2 are the numbers of non-zeros in u_1 and u_2, respectively. See [13] for the proof. Extensive simulations in [13] verify that the errors of (7) are usually within 5% when $k_1, k_2 > 20$.

We similarly derive the variances for $\hat{d}^{(2)}_{MF}$ and $\hat{d}^{(1)}_{MF}$. In a summary, we obtain (when $k_1 = k_2 = k$)

$$\text{Var}\left(\hat{a}_{MF}\right) = \text{E}\left(\frac{D}{D_s}\right)\left(\sum_{i=1}^{D} u_{1,i}^2 u_{2,i}^2 - \frac{a^2}{D}\right) \approx \frac{\max(f_1,f_2)}{D}\frac{1}{k}\left(D\sum_{i=1}^{D} u_{1,i}^2 u_{2,i}^2 - a^2\right), \tag{8}$$

$$\text{Var}\left(\hat{d}^{(2)}_{MF}\right) = \text{E}\left(\frac{D}{D_s}\right)\left(d^{(4)} - \frac{[d^{(2)}]^2}{D}\right) \approx \frac{\max(f_1,f_2)}{D}\frac{1}{k}\left(Dd^{(4)} - [d^{(2)}]^2\right), \tag{9}$$

$$\text{Var}\left(\hat{d}^{(1)}_{MF}\right) = \text{E}\left(\frac{D}{D_s}\right)\left(d^{(2)} - \frac{[d^{(1)}]^2}{D}\right) \approx \frac{\max(f_1,f_2)}{D}\frac{1}{k}\left(Dd^{(2)} - [d^{(1)}]^2\right). \tag{10}$$

where we denote $d^{(4)} = \sum_{i=1}^{D}\left(u_{1,i} - u_{2,i}\right)^4$.

The *sparsity* term $\frac{\max(f_1,f_2)}{D}$ reduces the variances significantly. If $\frac{\max(f_1,f_2)}{D}$ = 0.01, the variances can be reduced by a factor of 100, compared to *conventional random coordinate sampling*.

4 A Brief Introduction to Random Projections

We give a brief introduction to random projections, with which we compare CRS. (Normal) Random projections [17] are widely used in learning and data mining [2–4].

Random projections multiply the data matrix $\mathbf{A} \in \mathbb{R}^{n\times D}$ with a random matrix $\mathbf{R} \in \mathbb{R}^{D\times k}$ to generate a compact representation $\mathbf{B} = \mathbf{AR} \in \mathbb{R}^{n\times k}$. For estimating l_2 distances, $\mathbf{R}$ typically consists of i.i.d. entries in $N(0,1)$; hence we call it *normal random projections*. For l_1, $\mathbf{R}$ consists of i.i.d. Cauchy $C(0,1)$ [9]. However, the recent impossibility result [5] has ruled out estimators that could be metrics for dimension reduction in l_1.

Denote $v_1, v_2 \in \mathbb{R}^k$ the two rows in $\mathbf{B}$, corresponding to the original data points $u_1, u_2 \in \mathbb{R}^D$. We also introduce the notation for the marginal l_2 norms: $m_1 = \|u_1\|^2$, $m_2 = \|u_2\|^2$.

4.1 Normal Random Projections

In this case, $\mathbf{R}$ consists of i.i.d. $N(0,1)$. It is easy to show that the following linear estimators of the inner product a and the squared l_2 distance $d^{(2)}$ are unbiased

$$\hat{a}_{NRP,MF} = \frac{1}{k}v_1^{\text{T}}v_2, \qquad \hat{d}^{(2)}_{NRP,MF} = \frac{1}{k}\|v_1 - v_2\|^2, \tag{11}$$

with variances [15, 17]

$$\text{Var}\left(\hat{a}_{NRP,MF}\right) = \frac{1}{k}\left(m_1m_2 + a^2\right), \qquad \text{Var}\left(\hat{d}^{(2)}_{NRP,MF}\right) = \frac{2[d^{(2)}]^2}{k}. \tag{12}$$

Assuming that the margins $m_1 = \|u_1\|^2$ and $m_2 = \|u_2\|^2$ are known, [15] provides a maximum likelihood estimator, denoted by $\hat{a}_{NRP,MLE}$, whose (asymptotic) variance is

$$\text{Var}\left(\hat{a}_{NRP,MLE}\right) = \frac{1}{k}\frac{\left(m_1m_2 - a^2\right)^2}{m_1m_2 + a^2} + O(k^{-2}). \tag{13}$$

4.2 Cauchy Random Projections for Dimension Reduction in l_1

In this case, $\mathbf{R}$ consisting of i.i.d. entries in Cauchy $C(0,1)$. [9] proposed an estimator based on the absolute sample median. Recently, [14] proposed a variety of nonlinear estimators, including, a bias-corrected sample median estimator, a bias-corrected geometric mean estimator, and a bias-corrected

maximum likelihood estimator. An analog of the Johnson-Lindenstrauss (JL) lemma for dimension reduction in l_1 is also proved in [14], based on the bias-corrected geometric mean estimator.

We only list the maximum likelihood estimator derived in [14], because it is the most accurate one.

$$\hat{d}^{(1)}_{CRP,MLE,c} = \left(1 - \frac{1}{k}\right) \hat{d}^{(1)}_{CRP,MLE}, \tag{14}$$

where $\hat{d}^{(1)}_{CRP,MLE}$ solves a nonlinear MLE equation

$$-\frac{k}{\hat{d}^{(1)}_{CRP,MLE}} + \sum_{j=1}^{k} \frac{2\hat{d}^{(1)}_{CRP,MLE}}{(v_{1,j} - v_{2,j})^2 + \left(\hat{d}^{(1)}_{CRP,MLE}\right)^2} = 0. \tag{15}$$

[14] shows that

$$\text{Var}\left(\hat{d}^{(1)}_{CRP,MLE,c}\right) = \frac{2[d^{(1)}]^2}{k} + \frac{3[d^{(1)}]^2}{k^2} + O\left(\frac{1}{k^3}\right). \tag{16}$$

4.3 General Stable Random Projections for Dimension Reduction in l_p ($0 < p \leq 2$)

[10] generalized the bias-corrected geometric mean estimator to general stable random projections for dimension reduction in l_p ($0 < p \leq 2$), and provided the theoretical variances and exponential tail bounds. Of course, CRS can also be applied to approximating any l_p distances.

5 Improving CRS Using Marginal Information

It is often reasonable to assume that we know the marginal information such as marginal l_2 norms, numbers of non-zeros, or even marginal histograms. This often leads to (much) sharper estimates, by maximizing the likelihood under marginal constraints. In the boolean data case, we can express the MLE solution explicitly and derive a closed-form (asymptotic) variance. In general real-valued data, the joint likelihood is not available; we propose an approximate MLE solution.

5.1 Boolean (0/1) Data

In 0/1 data, estimating the inner product becomes estimating a two-way contingency table, which has four cells. Because of the margin constraints, there is only one degree of freedom. Therefore, it is not hard to show that the MLE of a is the solution, denoted by $\hat{a}_{0/1,MLE}$, to a cubic equation

$$\frac{s_{11}}{a} - \frac{s_{10}}{f_1 - a} - \frac{s_{01}}{f_2 - a} + \frac{s_{00}}{D - f_1 - f_2 + a} = 0, \tag{17}$$

where $s_{11} = \#\{j : \tilde{u}_{1,j} = \tilde{u}_{2,j} = 1\}$, $s_{10} = \#\{j : \tilde{u}_{1,j} = 1, \tilde{u}_{2,j} = 0\}$, $s_{01} = \#\{j : \tilde{u}_{1,j} = 0, \tilde{u}_{2,j} = 1\}$, $s_{00} = \#\{j : \tilde{u}_{1,j} = 0, \tilde{u}_{2,j} = 0\}$, $j = 1, 2, ..., D_s$.

The (asymptotic) variance of $\hat{a}_{0/1,MLE}$ is proved [11–13] to be

$$\text{Var}(\hat{a}_{0/1,MLE}) = \text{E}\left(\frac{D}{D_s}\right) \frac{1}{\frac{1}{a} + \frac{1}{f_1 - a} + \frac{1}{f_2 - a} + \frac{1}{D - f_1 - f_2 + a}}. \tag{18}$$

5.2 Real-valued Data

A practical solution is to assume some parametric form of the (bivariate) data distribution based on prior knowledge; and then solve an MLE considering various constraints. Suppose the samples $(\tilde{u}_{1,j},\ \tilde{u}_{2,j})$ are i.i.d. bivariate normal with moments determined by the population moments, i.e.,

$$\begin{bmatrix} \tilde{v}_{1,j} \\ \tilde{v}_{2,j} \end{bmatrix} = \begin{bmatrix} \tilde{u}_{1,j} - \bar{u}_1 \\ \tilde{u}_{2,j} - \bar{u}_2 \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tilde{\Sigma}\right), \tag{19}$$

$$\tilde{\Sigma} = \frac{1}{D_s}\frac{D_s}{D}\begin{bmatrix} \|u_1\|^2 - D\bar{u}_1^2 & u_1^{\text{T}}u_2 - D\bar{u}_1\bar{u}_2 \\ u_1^{\text{T}}u_2 - D\bar{u}_1\bar{u}_2 & \|u_2\|^2 - D\bar{u}_2^2 \end{bmatrix} = \frac{1}{D_s}\begin{bmatrix} \ddot{m}_1 & \ddot{a} \\ \ddot{a} & \ddot{m}_2 \end{bmatrix}, \tag{20}$$

where $\bar{u}_1 = \sum_{i=1}^{D} u_{1,i}/D$, $\bar{u}_2 = \sum_{i=1}^{D} u_{2,i}/D$ are the population means. $\ddot{m}_1 = \frac{D_s}{D}\left(\|u_1\|^2 - D\bar{u}_1^2\right)$, $\ddot{m}_2 = \frac{D_s}{D}\left(\|u_2\|^2 - D\bar{u}2^2\right)$, $\ddot{a} = \frac{D_s}{D}\left(u_1^{\text{T}}u_2 - D\bar{u}_1\bar{u}_2\right)$. Suppose that $\bar{u}_1$, $\bar{u}_2$, $m_1 = \|u_1\|^2$ and $m_2 = \|u_2\|^2$ are known, an MLE for $a = u_1^{\text{T}}u_2$, denoted by $\hat{a}_{MLE,N}$, is

$$\hat{a}_{MLE,N} = \frac{D}{D_s}\hat{\ddot{a}} + D\bar{u}_1\bar{u}_2, \tag{21}$$

where, similar to Lemma 2 of [15], $\hat{\ddot{a}}$ is the solution to a cubic equation:

$$\ddot{a}^3 - \ddot{a}^2\left(\tilde{v}_1^{\mathrm{T}}\tilde{v}_2\right) + \ddot{a}\left(-\ddot{m}_1\ddot{m}_2 + \ddot{m}_1\|\tilde{v}_2\|^2 + \ddot{m}_2\|\tilde{v}_1\|^2\right) - \ddot{m}_1\ddot{m}_2\tilde{v}_1^{\mathrm{T}}\tilde{v}_2 = 0. \tag{22}$$

$\hat{a}_{MLE,N}$ is fairly robust, although sometimes we observe the biases are quite noticeable. In general, this is a good bias-variance trade-off (especially when k is not too large). Intuitively, the reason why this (seemly crude) assumption of bivariate normality works well is because, once we have fixed the margins, we have removed to a large extent the non-normal component of the data.

6 Theoretical Comparisons of CRS With Random Projections

As reflected by their variances, for general data types, whether CRS is better than random projections depends on two competing factors: data sparsity and data heavy-tailedness. However, in the following two important scenarios, CRS outperforms random projections.

6.1 Boolean (0/1) data

In this case, the marginal norms are the same as the numbers of non-zeros, i.e., $m_i = \|u_i\|^2 = f_i$.

Figure 3 plots the ratio, $\frac{\mathrm{Var}(\hat{a}_{MF})}{\mathrm{Var}(\hat{a}_{NRP,MF})}$, verifying that CRS is (considerably) more accurate:

$$\frac{\mathrm{Var}\left(\hat{a}_{MF}\right)}{\mathrm{Var}\left(\hat{a}_{NRP,MF}\right)} = \frac{\max(f_1,f_2)}{f_1f_2+a^2}\,\frac{1}{\frac{1}{a}+\frac{1}{D-a}} \leq \frac{\max(f_1,f_2)a}{f_1f_2+a^2} \leq 1.$$

Figure 4 plots $\frac{\mathrm{Var}\left(\hat{a}_{0/1,MLE}\right)}{\mathrm{Var}(\hat{a}_{NRP,MLE})}$. In most possible range of the data, this ratio is less than 1. When u_1 and u_2 are very close (e.g., $a \approx f_2 \approx f_1$), random projections appear more accurate. However, when this does occur, the absolute variances are so small (even zero) that their ratio does not matter.

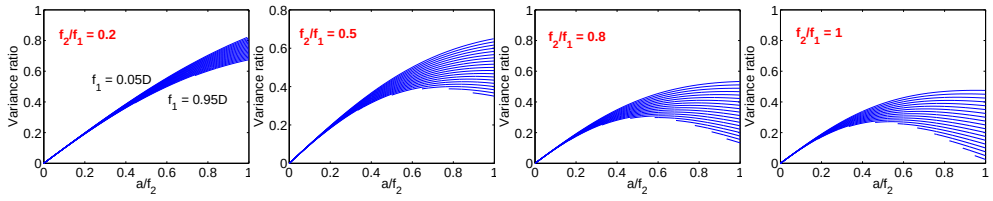


Figure 3: The variance ratios, $\frac{\mathrm{Var}(\hat{a}_{MF})}{\mathrm{Var}(\hat{a}_{NRP,MF})}$, show that CRS has smaller variances than random projections, when no marginal information is used. We let $f_1 \geq f_2$ and $f_2 = \alpha f_1$ with $\alpha = 0.2, 0.5, 0.8, 1.0$. For each α, we plot from $f_1 = 0.05D$ to $f_1 = 0.95D$ spaced at $0.05D$.

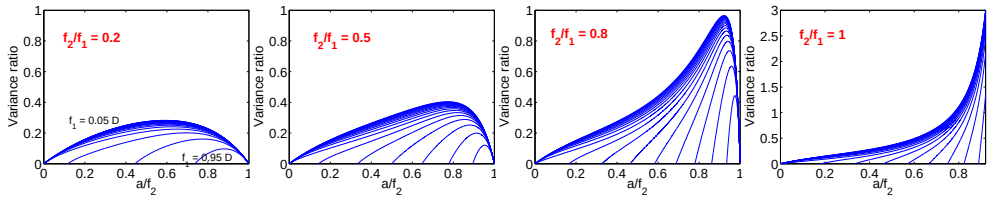


Figure 4: The ratios, $\frac{\mathrm{Var}\left(\hat{a}_{0/1,MLE}\right)}{\mathrm{Var}(\hat{a}_{NRP,MLE})}$, show that CRS usually has smaller variances than random projections, except when $f_1 \approx f_2 \approx a$.

6.2 Nearly Independent Data

Suppose two data points u_1 and u_2 are independent (or less strictly, uncorrelated to the second order), it is easy to show that the variance of CRS is always smaller:

$$\mathrm{Var}\left(\hat{a}_{MF}\right) \leq \frac{\max(f_1,f_2)}{D}\frac{m_1m_2}{k} \leq \mathrm{Var}\left(\hat{a}_{NRP,MF}\right) = \frac{m_1m_2+a^2}{k}, \tag{23}$$

even if we ignore the data sparsity. Therefore, CRS will be much better for estimating inner products in nearly independent data. Once we have obtained the inner products, we can infer the l_2 distances easily by $d^{(2)} = m_1 + m_2 - 2a$, since the margins, m_1 and m_2, are easy to obtain exactly.

In high dimensions, it is often the case that most of the data points are only very weakly correlated.

6.3 Comparing the Computational Efficiency

As previously mentioned, the cost of constructing sketches for $\mathbf{A} \in \mathbb{R}^{n \times D}$ would be $O(nD)$ (or more precisely, $O(\sum_{i=1}^{n} f_i)$). The cost of (normal) random projections would be $O(nDk)$, which can be reduced to $O(nDk/3)$ using *sparse random projections* [1]. Therefore, it is possible that CRS is considerably more efficient than random projections in the sampling stage.[2]

In the estimation stage, CRS costs $O(2k)$ to compute the sample distance for each pair. This cost is only $O(k)$ in random projections. Since k is very small, the difference should not be a concern.

7 Empirical Evaluations

We compare CRS with random projections (RP) using real data, including $n = 100$ randomly sampled documents from the NSF data [7] (sparsity $\approx 1\%$), $n = 100$ documents from the NEWSGROUP data [4] (sparsity $\approx 1\%$), and one class of the COREL image data ($n = 80$, sparsity $\approx 5\%$). We estimate all pairwise inner products, l_1 and l_2 distances, using both CRS and RP. For each pair, we obtain 50 runs and average the absolute errors. We compare the median errors and the percentage in which CRS does better than random projections.

The results are presented in Figures 5, 6, 7. In each panel, the dashed curve indicates that we sample each data point with equal sample size (k). For CRS, we can adjust the sample size according to the sparsity, reflected by the solid curves. We adjust sample sizes only roughly. The data points are divided into 3 groups according to sparsity. Data in different groups are assigned different sample sizes for CRS. For random projections, we use the average sample size.

For both NSF and NEWSGROUP data, CRS overwhelmingly outperforms RP for estimating inner products and l_2 distances (both using the marginal information). CRS also outperforms RP for approximating l_1 and l_2 distances (without using the margins).

For the COREL data, CRS still outperforms RP for approximating inner products and l_2 distances (using the margins). However, RP considerably outperforms CRS for approximating l_1 distances and l_2 distances (without using the margins). Note that the COREL image data are not too sparse and are considerably more heavy-tailed than the NSF and NEWSGROUP data [13].

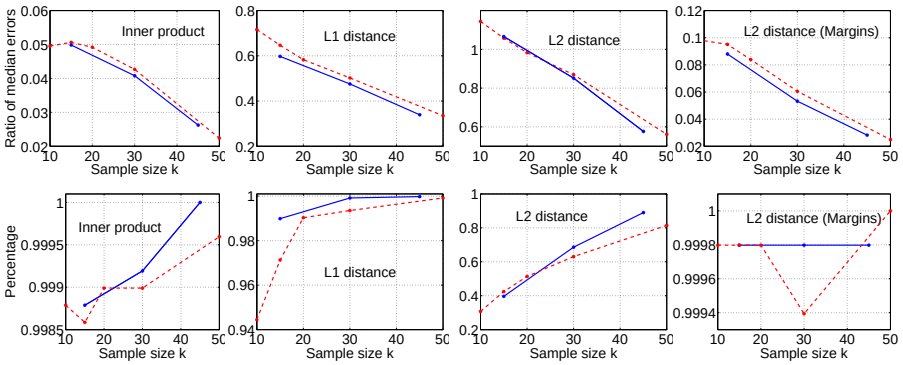


Figure 5: NSF data. Upper four panels: ratios (CRS over RP (random projections)) of the median absolute errors; values < 1 indicate that CRS does better. Bottom four panels: percentage of pairs for which CRS has smaller errors than RP; values > 0.5 indicate that CRS does better. Dashed curves correspond to fixed sample sizes while solid curves indicate that we (crudely) adjust sketch sizes in CRS according to data sparsity. In this case, CRS is overwhelmingly better than RP for approximating inner products and l_2 distances (both using margins).

8 Conclusion

There are many applications of l_1 and l_2 distances on large sparse datasets. We propose a new sketch-based method, *Conditional Random Sampling (CRS)*, which is provably better than random projections, at least for the important special cases of boolean data and nearly independent data. In general non-boolean data, CRS compares favorably, both theoretically and empirically, especially when we take advantage of the margins (which are easier to compute than distances).

[2] [16] proposed *very sparse random projections* to reduce the cost $O(nDk)$ down to $O(n\sqrt{D}k)$.

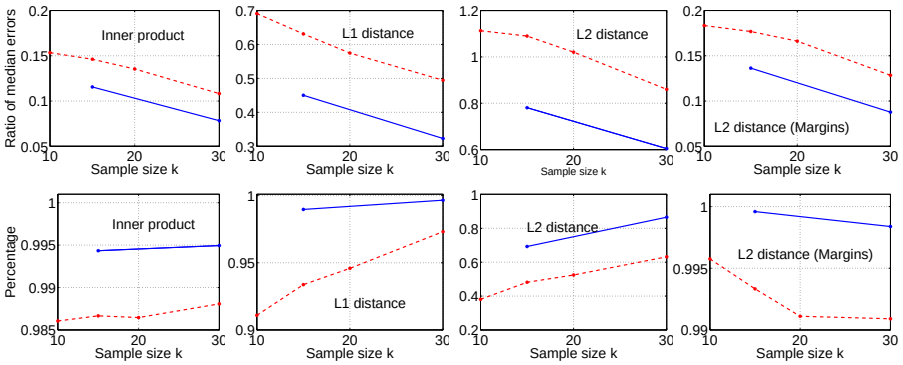


Figure 6: NEWSGROUP data. The results are quite similar to those in Figure 5 for the NSF data. In this case, it is more obvious that adjusting sketch sizes helps CRS.

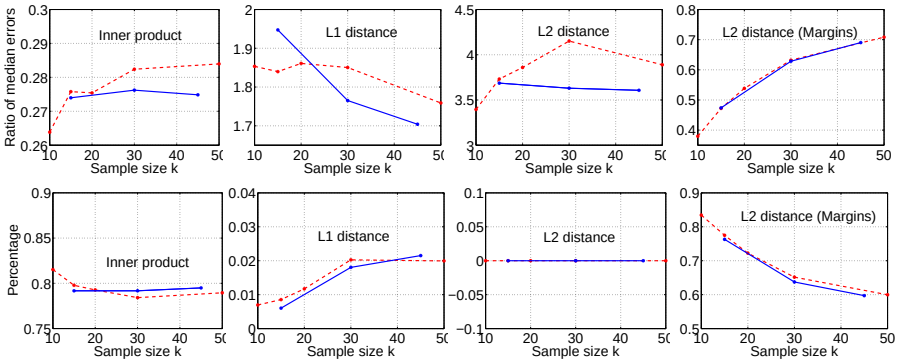


Figure 7: COREL image data.

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